Lecture 19, November 14

- **Line integrals.** The integral of $f(x,y)$ over a curve C in the xy-plane is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot ||\boldsymbol{r}'(t)||\,dt,$$

where $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ is the equation of the curve and $a \leq t \leq b$. The integrals

$$\int_C f(x,y)\,dx, \qquad \int_C f(x,y)\,dy, \qquad \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

are defined similarly in terms of $dx = x'(t)\,dt$, $dy = y'(t)\,dt$ and $d\boldsymbol{r} = \boldsymbol{r}'(t)\,dt$.

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Example 1. Take $f(x,y) = xy^2$ and let C be the line from $(0,0)$ to $(1,2)$. Then

$$\boldsymbol{r}(t) = \langle t, 2t\rangle \quad \Longrightarrow \quad \boldsymbol{r}'(t) = \langle 1, 2\rangle \quad \Longrightarrow \quad ||\boldsymbol{r}'(t)|| = \sqrt{5}$$

and we have $0 \leq t \leq 1$. Since $x = t$ and $y = 2t$ throughout the curve, we find that

$$\int_C xy^2\,ds = \int_0^1 t(2t)^2 \cdot \sqrt{5}\,dt = 4\sqrt{5}\int_0^1 t^3\,dt = \sqrt{5}.$$

Example 2. Take $f(x,y) = x$ and let C be the part of the unit circle that lies in the first quadrant. Then we have $\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle$ with $0 \leq t \leq \pi/2$ and so

$$\boldsymbol{r}'(t) = \langle -\sin t, \cos t\rangle \quad \Longrightarrow \quad ||\boldsymbol{r}'(t)|| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

Since $x = \cos t$ by above, we conclude that

$$\int_C x\,ds = \int_0^{\pi/2} \cos t\,dt = \sin(\pi/2) - \sin 0 = 1.$$

Example 3. Take $\boldsymbol{F}(x,y) = \langle y, x\rangle$ and let C be as in the previous example. Then

$$\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle \quad \Longrightarrow \quad d\boldsymbol{r} = \langle -\sin t, \cos t\rangle\ dt$$

and also $\boldsymbol{F} = \langle y, x\rangle = \langle \sin t, \cos t\rangle$ throughout the curve, so we get

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{\pi/2} \left(-\sin^2 t + \cos^2 t\right)dt = \int_0^{\pi/2} \cos(2t)\,dt = \left[\frac{\sin(2t)}{2}\right]_0^{\pi/2} = 0.$$