## Lecture 7, October 8

- **Chain rule.** Suppose that $f(x,y)$ depends on two variables, each of which depends on a third variable $t$. Then the derivative $f_t$ is the sum of two terms, namely

$$f_t = f_x x_t + f_y y_t.$$

Similar formulas hold for functions of three or more variables; for instance,

$$f = f(x,y,z) \quad \Longrightarrow \quad f_t = f_x x_t + f_y y_t + f_z z_t.$$

- **Implicit differentiation.** Let $F(x,y,z) = 0$ be a relation between three variables. If we view $z$ as a function of $x$ and $y$, then its partial derivatives are

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Example 1.** Let $z = x^2 y$, where $x = \sin t + 2s$ and $y = e^t + s^2 t$. Then we have

$$z_t = z_x x_t + z_y y_t = 2xy \cdot \cos t + x^2 \cdot (e^t + s^2),$$
$$z_s = z_x x_s + z_y y_s = 2xy \cdot 2 + x^2 \cdot 2st.$$

**Example 2.** Let $z = e^{xy}$, where $x = u/v$ and $y = u^2 + 3v$. In this case,

$$z_u = z_x x_u + z_y y_u = ye^{xy} \cdot (1/v) + xe^{xy} \cdot 2u,$$
$$z_v = z_x x_v + z_y y_v = ye^{xy} \cdot (-u/v^2) + xe^{xy} \cdot 3.$$

**Example 3.** Let $w = x^2 y z^3$, where $x = 1 + t^2$, $y = 2 - t$ and $z = 2 - t^3$. Then

$$\begin{aligned} w_t &= w_x x_t + w_y y_t + w_z z_t \\ &= 2xyz^3 \cdot 2t + x^2 z^3 \cdot (-1) + 3x^2 y z^2 \cdot (-3t^2). \end{aligned}$$

At time $t = 1$, for instance, we have $x = 2$ and $y = z = 1$, so

$$w_t = 4xyz^3 - x^2 z^3 - 9x^2 y z^2 = 8 - 4 - 36 = -32.$$

**Example 4.** Suppose $x, y, z$ are related by the formula $xy^2 + xz^2 + yz = 0$. Then

$$z_x = -\frac{(xy^2 + xz^2 + yz)_x}{(xy^2 + xz^2 + yz)_z} = -\frac{y^2 + z^2}{2xz + y},$$
$$z_y = -\frac{(xy^2 + xz^2 + yz)_y}{(xy^2 + xz^2 + yz)_z} = -\frac{2xy + z}{2xz + y}.$$