Lecture 6, October 5

- **Partial derivatives.** Given a function $f(x, y)$ of two variables, we define f_x to be its derivative with respect to x when y is treated as a constant. The partial derivative f_x gives the rate at which f is changing in the x-direction. The partial derivative f_y is defined similarly by differentiating with respect to y, while treating x as a constant.
- **Mixed partials.** If the mixed partial derivatives f_{xy} and f_{yx} are continuous, then they must be equal to one another. This is the case for all standard functions.

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Example 1. Let $f(x, y) = x^2y$. Then $f_x = 2xy$ and $f_y = x^2$. The mixed partials are

$$f_{xy} = (2xy)_y = 2x, \qquad f_{yx} = (x^2)_x = 2x.$$

Example 2. Let $f(x, y) = xy^2 + 2xy + y^2$. Then we have

$$f_x = y^2 + 2y, \qquad f_y = 2xy + 2x + 2y, \qquad f_{xy} = f_{yx} = 2y + 2.$$

Example 3. Let $f(x, y) = \sin(x^2y)$. Using the chain rule, one finds that

$$\begin{aligned} f_x(x, y) &= \cos(x^2y) \cdot (x^2y)_x = \cos(x^2y) \cdot 2xy, \\ f_y(x, y) &= \cos(x^2y) \cdot (x^2y)_y = \cos(x^2y) \cdot x^2. \end{aligned}$$

Example 4. Let $f(x, y) = y \sin(xy)$. To compute f_x, we argue as before to get

$$f_x(x, y) = y \cos(xy) \cdot (xy)_x = y^2 \cos(xy).$$

To compute f_y, however, one needs to resort to the product rule; this gives

$$f_y(x, y) = \sin(xy) + y \cos(xy) \cdot (xy)_y = \sin(xy) + xy \cos(xy).$$

Example 5. Let $f(x, y, z) = (x^3 + 2y^2 + 3z)^4$. Then the first-order derivatives are

$$\begin{aligned} f_x(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 3x^2 = 12x^2(x^3 + 2y^2 + 3z)^3, \\ f_y(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 4y = 16y(x^3 + 2y^2 + 3z)^3, \\ f_z(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 3 = 12(x^3 + 2y^2 + 3z)^3. \end{aligned}$$

In this case, equality of mixed partials means that

$$f_{xy} = f_{yx}, \qquad f_{xz} = f_{zx}, \qquad f_{yz} = f_{zy}.$$