Lecture 16, October 31

- **Triple integrals in cylindrical coordinates.** One has the formulas

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2+y^2=r^2, \qquad dV = r\,dz\,dr\,d\theta.$$

- **Triple integrals in spherical coordinates.** One has the formulas

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi, \qquad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

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Example 1. Consider the solid G which is bounded by the cone $z = \sqrt{x^2+y^2}$ from below and by the sphere $x^2+y^2+z^2=8$ from above. To find its projection R onto the xy-plane, we find the points at which the cone meets the sphere, namely

$$x^2+y^2+z^2=8 \quad\Longrightarrow\quad x^2+y^2+(x^2+y^2)=8 \quad\Longrightarrow\quad x^2+y^2=4.$$

This gives a circle of radius 2 in the xy-plane, so the volume of G is

$$\text{Volume} = \iiint_G dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx.$$

In terms of cylindrical coordinates, the circle $x^2+y^2=4$ becomes $r^2=4$ and so

$$\text{Volume} = \int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta.$$

In terms of spherical coordinates, finally, the equation of the cone is

$$z=\sqrt{x^2+y^2}=r \quad\Longrightarrow\quad \tan\phi=\frac{r}{z}=1 \quad\Longrightarrow\quad \phi=\pi/4$$

and we can compute the volume of the solid as

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3\sin\phi}{3}\right]_{\rho=0}^{\sqrt{8}} d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\frac{16\sqrt{2}}{3}\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[-\frac{16\sqrt{2}}{3}\cos\phi\right]_{\phi=0}^{\pi/4} d\theta \\
&= \int_0^{2\pi}\frac{16\sqrt{2}}{3}\left(1-\frac{\sqrt{2}}{2}\right)d\theta = \frac{32\pi}{3}\cdot(\sqrt{2}-1).
\end{aligned}$$