Lecture 15, October 26

- **Laminas.** If a lamina R has density function $\delta(x,y)$, then its mass is given by

$$M = \iint_R \delta(x,y)\,dA,$$

while its center of gravity is the point (x_0, y_0) whose coordinates are

$$x_0 = \frac{1}{M}\iint_R x\delta(x,y)\,dA, \qquad y_0 = \frac{1}{M}\iint_R y\delta(x,y)\,dA.$$

For laminas of constant density, the center of gravity is also known as the centroid.

- **Triple integrals.** Suppose that G is a solid which is bounded above by $z = g(x,y)$ and below by $z = h(x,y)$. If its projection onto the xy-plane is the region R, then

$$\text{Volume of } G = \iiint_G dV = \iint_R [g(x,y) - h(x,y)]\,dA.$$

. . . .

Example 1. The lamina R inside the unit circle with $\delta(x,y) = x^2 + y^2$ has mass

$$M = \iint_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Its center of gravity (x_0, y_0) should be the origin by symmetry. In fact, we have

$$x_0 = \frac{2}{\pi}\iint_R x(x^2+y^2)\,dA = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 r^4\cos\theta\,dr\,d\theta = \frac{2}{\pi}\int_0^{2\pi}\frac{\cos\theta}{5}\,d\theta = 0$$

and a similar computation gives $y_0 = 0$ as well.

Example 2. Let G be the solid which is bounded by $z = x^2 + y^2$ from below and by the plane $z = 1$ from above. Its projection R onto the xy-plane is $x^2 + y^2 = 1$, so

$$\begin{aligned}\text{Volume of } G &= \iint_R (1 - x^2 - y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\cdot r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right)d\theta = \frac{\pi}{2}.\end{aligned}$$