Lecture 14, October 24

- **Parametric surfaces.** A surface in $\mathbb{R}^3$ can be described using a vector equation
$$\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle .$$
Its normal vector is given by the cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v$, while the area of the surface which lies above the region R in the uv-plane is
$$\text{Surface area} = \iint_R ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, dA.$$

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Example 1. The parametric equation of the cone $z = \sqrt{x^2+y^2}$ is given by
$$\boldsymbol{r} = \langle x,y,z\rangle = \left\langle x, y, \sqrt{x^2+y^2}\right\rangle = \langle r\cos\theta, r\sin\theta, r\rangle .$$

Example 2. We compute the area of the part of the cylinder $x^2+y^2=1$ which lies between the planes $z=0$ and $z=3$. Its parametric equation is
$$\boldsymbol{r} = \langle \cos\theta, \sin\theta, z\rangle , \qquad 0\le\theta\le 2\pi, \qquad 0\le z\le 3.$$
Since the normal vector is given by
$$\boldsymbol{r}_\theta \times \boldsymbol{r}_z = \langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0,0,1\rangle = \langle \cos\theta, \sin\theta, 0\rangle ,$$
its length is equal to 1 and so the area of the cylinder is
$$\int_0^{2\pi}\int_0^3 ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z||\, dz\, d\theta = \int_0^{2\pi}\int_0^3 dz\, d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi.$$

Example 3. We compute the area of the part of the cone $z=\sqrt{x^2+y^2}$ which lies inside the cylinder $x^2+y^2=4$. Its parametric equation is
$$\boldsymbol{r} = \langle x,y,z\rangle = \langle r\cos\theta, r\sin\theta, r\rangle , \qquad 0\le\theta\le 2\pi, \qquad 0\le r\le 2.$$
To find its area, we first compute the normal vector
$$\boldsymbol{r}_r \times \boldsymbol{r}_\theta = \langle \cos\theta, \sin\theta, 1\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle = \langle -r\cos\theta, -r\sin\theta, r\rangle .$$
This gives $||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| = \sqrt{2r^2} = r\sqrt{2}$, so the area of the cone is
$$\int_0^{2\pi}\int_0^2 ||\boldsymbol{r}_r \times \boldsymbol{r}_\theta||\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{2}\, dr\, d\theta = \int_0^{2\pi} 2\sqrt{2}\, d\theta = 4\pi\sqrt{2}.$$