Lecture 13, October 22

- **Polar coordinates.** Expressing a double integral in polar coordinates, one has

$$\iint_R f(x,y)\,dA = \int\int f(r\cos\theta, r\sin\theta)\cdot r dr\, d\theta$$

for some suitable limits of integration that describe the region R.

. . . .

Example 1. If R is the region depicted on the left side of the figure, then

$$\iint_R (x^2+y^2)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

Expressing this integral in polar coordinates, one can also write it as

$$\iint_R (x^2+y^2)\,dA = \int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta.$$

Example 2. We use polar coordinates in order to compute the integral

$$I = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy.$$

In this case, the region of integration is bounded by the line $x = y$ on the left and by the circle $x = \sqrt{4-y^2}$ on the right. Note that these two intersect when

$$y = \sqrt{4-y^2} \quad\Longrightarrow\quad y^2 = 4-y^2 \quad\Longrightarrow\quad 2y^2 = 4 \quad\Longrightarrow\quad y^2 = 2.$$

This explains the upper limit of integration $y = \sqrt{2}$. The region of integration is thus the one depicted on the right and we can describe it using polar coordinates to get

$$I = \int_0^{\pi/4} \int_0^2 r\cdot r\,dr\,d\theta = \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^2 d\theta = \int_0^{\pi/4} \frac{8}{3}\,d\theta = \frac{2\pi}{3}.$$

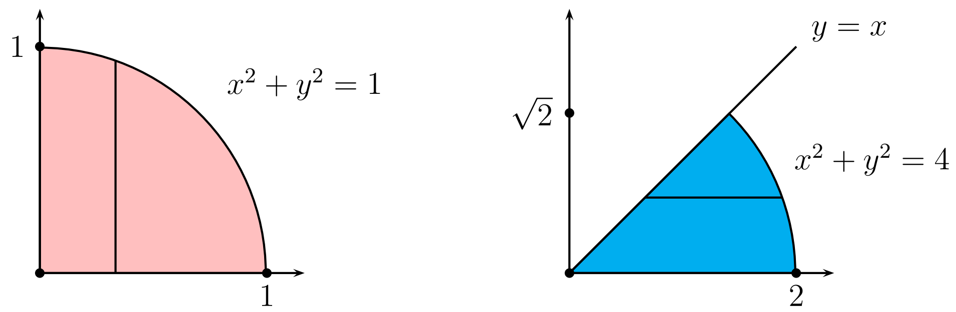

Figure: The regions of integration for Examples 1 and 2.