Lecture 11, October 17

- **Double integrals.** The double integral of $f(x,y)$ over a region R in the xy-plane is defined in terms of Riemann sums as

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

If f is positive, then this expression gives the volume of the solid that lies below the graph of f and above the region R in the xy-plane. When $R = [a,b] \times [c,d]$, one has

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

. . . .

Example 1. Consider $f(x,y) = x^2 y$ over the rectangle $R = [0,2] \times [0,1]$. Then

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^2 x^2 y\,dx\,dy$$

and we can focus on the inner integral first. Integrating with respect to x, we get

$$\int_0^2 x^2 y\,dx = y \int_0^2 x^2\,dx = y \left[\frac{x^3}{3}\right]_{x=0}^{2} = \frac{8y}{3}$$

and so the double integral is

$$\iint_R f(x,y)\,dA = \int_0^1 \frac{8y}{3}\,dy = \left[\frac{4y^2}{3}\right]_{y=0}^{1} = \frac{4}{3}.$$

Alternatively, one may reach the same answer by writing

$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^1 x^2 y\,dy\,dx$$

and by integrating with respect to y first. This approach gives

$$\int_0^1 x^2 y\,dy = x^2 \int_0^1 y\,dy = x^2 \left[\frac{y^2}{2}\right]_{y=0}^{1} = \frac{x^2}{2}$$

for the inner integral, so the double integral is equal to

$$\iint_R f(x,y)\,dA = \int_0^2 \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_{x=0}^{2} = \frac{4}{3}.$$