Lecture 8, October 10

- **Directional derivative.** We denote by $D_{\boldsymbol{u}} f(x_0, y_0)$ the rate at which the function f changes at the point (x_0, y_0) in the direction of the unit vector $\boldsymbol{u} = \langle a, b\rangle$; that is,

$$D_{\boldsymbol{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0).$$

If the vector $\boldsymbol{u}$ does not have unit length, one may simply divide it by its length.

- **Gradient vector.** Given a function $f(x, y)$ of two variables, we define its gradient to be the vector $\nabla f(x, y) = \langle f_x, f_y\rangle$. Using this notation, one can write

$$D_{\boldsymbol{u}} f(x_0, y_0) = \boldsymbol{u} \cdot \nabla f(x_0, y_0).$$

The gradient vector ∇f gives the direction of most rapid increase at each point and the rate of change in that direction is $||\nabla f||$. Similarly, $-\nabla f$ gives the direction of most rapid decrease at each point and the rate of change in that direction is $-||\nabla f||$.

- **Functions of more variables.** When $f = f(x, y, z)$, for instance, one has

$$\nabla f = \langle f_x, f_y, f_z\rangle\,, \qquad D_{\boldsymbol{u}} f = \boldsymbol{u} \cdot \nabla f$$

and the gradient vector ∇f has the exact same interpretation as before.

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Example 1. Let $f(x, y) = 3x^2 - 4xy^2$. When it comes to the point $(1, 1)$, we have

$$\nabla f = \langle f_x, f_y\rangle = \left\langle 6x - 4y^2, -8xy\right\rangle = \langle 2, -8\rangle\,.$$

Thus, the directional derivative of f in the direction of $\boldsymbol{u} = \langle 3/5, 4/5\rangle$ is

$$D_{\boldsymbol{u}} f = \boldsymbol{u} \cdot \nabla f = \frac{6}{5} - \frac{32}{5} = -\frac{26}{5}.$$

Example 2. Consider the function $f(x, y, z) = xyz^2$ at the point $(1, 2, 1)$. Then

$$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle yz^2, xz^2, 2xyz\right\rangle = \langle 2, 1, 4\rangle$$

gives the direction of most rapid increase and the corresponding rate of change is

$$||\nabla f|| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}.$$

To find the rate of change in the direction of $\boldsymbol{v} = \langle 2, 1, 2\rangle$, we note that

$$||\boldsymbol{v}|| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3,$$

so $\boldsymbol{v}$ is not a unit vector. Since $\boldsymbol{w} = \frac{1}{3}\boldsymbol{v} = \langle 2/3, 1/3, 2/3\rangle$ is a unit vector, we get

$$D_{\boldsymbol{v}} f = D_{\boldsymbol{w}} f = \boldsymbol{w} \cdot \nabla f = \frac{4}{3} + \frac{1}{3} + \frac{8}{3} = \frac{13}{3}.$$