Lecture 4, October 1

- **Cross product.** The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\boldsymbol{v} \times \boldsymbol{w} = \langle v_2 w_3 - v_3 w_2,\, v_3 w_1 - v_1 w_3,\, v_1 w_2 - v_2 w_1 \rangle .$$

This vector is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$, while its length is

$$||\boldsymbol{v} \times \boldsymbol{w}|| = ||\boldsymbol{v}|| \cdot ||\boldsymbol{w}|| \cdot \sin\theta.$$

- **Normal vector.** We say that the vector $\boldsymbol{n}$ is normal to a plane, if $\boldsymbol{n}$ is orthogonal to every vector that lies on the plane. If a plane passes through $A(a_1, a_2, a_3)$ and its normal vector is $\boldsymbol{n} = \langle n_1, n_2, n_3 \rangle$, then its equation is

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0.$$

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Example 1. Consider the points $A(2, 3, 4)$, $B(1, 0, 2)$ and $C(3, 2, 1)$. To find the plane that passes through these points, we note that its normal vector is

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -1, -3, -2 \rangle \times \langle 1, -1, -3 \rangle = \langle 7, -5, 4 \rangle .$$

Since the plane passes through $A(2, 3, 4)$, its equation is then

$$7(x - 2) - 5(y - 3) + 4(z - 4) = 0 \quad\Longrightarrow\quad 7x - 5y + 4z = 15.$$

Example 2. Consider the line through $P(2, 4, 1)$ and $Q(4, 1, 5)$. To find the point at which it intersects the plane $x - 2y + 3z = 37$, we first find the equation of the line. Since $\overrightarrow{PQ} = \langle 2, -3, 4 \rangle$ is the direction of the line, its equation is

$$x = 2 + 2t, \qquad y = 4 - 3t, \qquad z = 1 + 4t.$$

The point we wish to find is the point which satisfies this equation (because it is on the line) as well as $x - 2y + 3z = 37$ (because it is on the plane). This gives

$$(2 + 2t) - 2(4 - 3t) + 3(1 + 4t) = 37 \quad\Longrightarrow\quad 20t = 40 \quad\Longrightarrow\quad t = 2,$$

so the point of intersection is the point

$$(x, y, z) = (2 + 2t, 4 - 3t, 1 + 4t) = (6, -2, 9).$$