**MA2327, Sample exam**
(Answer 3 of the 4 problems)

1. (a) [4 points] Determine the unique solution of the initial value problem

$$y'(x) = 2x \cdot (y(x)+1)^2, \qquad y(0) = 2.$$

(b) [6 points] Solve the first-order linear equation

$$xy'(x) - y(x) = x^3 e^x, \qquad x > 0.$$

(c) [10 points] Use the substitution $z = \ln y(x)$ to find all positive solutions of

$$xy'(x) - 4x^2 y(x) + 2y(x) \ln y(x) = 0, \qquad x > 0.$$

2. (a) [6 points] Let $x_0, y_0 \in \mathbb{R}$ be given and consider the initial value problem

$$y'(x) = 2x + (y(x) - x^2)^{2/3}, \qquad y(x_0) = y_0.$$

For which values of $x_0, y_0$ is the solution unique? What can you say about the remaining values? Hint: an obvious change of variables gives $z'(x) = z(x)^{2/3}$.

(b) [6 points] Find all continuous functions $y(x)$ that satisfy the integral equation

$$y(x) = x^2 + \int_0^x \frac{y(s)}{s+1}\, ds.$$

(c) [8 points] Let $a > 0$ be given and consider the initial value problem

$$y'(x) = \frac{x^2+1}{x^2+x} \cdot y(x) \cdot \sin y(x), \qquad y(1) = a.$$

Use the associated integral equation to find a constant $C > 0$ such that

$$|y(x)| \le \frac{Cxe^x}{(x+1)^2} \quad \text{for all } x \ge 1.$$

3. (a) [10 points] Solve the third-order linear inhomogeneous equation

$$y'''(t) - 4y''(t) + 5y'(t) - 2y(t) = e^{2t} - \sin t.$$

   (b) [10 points] It is easy to check that $y_1(t) = e^{2t}$ satisfies the linear equation

$$ty''(t) - (4t+2)y'(t) + (4t+4)y(t) = 0, \qquad t > 0.$$

   Use reduction of order to find a basis of solutions for this equation.

4. (a) [10 points] Let $a \in \mathbb{R}$ be fixed and consider the linear system

$$x'(t) = ax(t) + y(t), \qquad y'(t) = x(t) + ay(t).$$

   For which values of $a$ is the zero solution stable? Asymptotically stable?

   (b) [4 points] Show that the zero solution is an asymptotically stable solution of

$$x'(t) = -x - xy^2, \qquad y'(t) = -y - x^2 y$$

   by finding a strict Lyapunov function of the form $V(x, y) = ax^2 + by^2$.

   (c) [6 points] Let $a \in \mathbb{R}$ be fixed and consider the linear system

$$x'(t) = -x(t) + y(t), \qquad y'(t) = x(t) - ay(t).$$

   For which values of $a$ is $V(x, y) = x^2 + y^2$ a strict Lyapunov function?

**MA2327, Sample exam solutions**

**1a.** Since $y = -1$ is a solution, every other solution satisfies $y \neq -1$ at all points, so

$$\frac{dy}{dx} = 2x(y+1)^2 \quad \Longrightarrow \quad \int \frac{dy}{(y+1)^2} = \int 2x\,dx \quad \Longrightarrow \quad -\frac{1}{y+1} = x^2 + C.$$

The initial condition $y(0) = 2$ implies that $-1/3 = C$ and this finally gives

$$y + 1 = -\frac{1}{x^2 + C} = -\frac{3}{3x^2 - 1} \quad \Longrightarrow \quad y = -\frac{3x^2 + 2}{3x^2 - 1}.$$

**1b.** The standard form is $y'(x) - \frac{1}{x}\,y(x) = x^2 e^x$, so an integrating factor is

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = e^{-\ln x + C} = Kx^{-1}.$$

Letting $\mu(x) = x^{-1}$ for simplicity, we must thus have

$$\Big[\mu(x)y(x)\Big]' = xe^x \quad \Longrightarrow \quad x^{-1}y(x) = \int xe^x\,dx.$$

To compute the integral, one needs to integrate by parts to get

$$\int xe^x\,dx = \int x(e^x)'\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Once we now combine the last two equations, we may conclude that

$$\frac{y(x)}{x} = (x-1)e^x + C \quad \Longrightarrow \quad y(x) = (x^2 - x)e^x + Cx.$$

**1c.** If $y(x)$ is a positive solution of the given equation, then $z(x) = \ln y(x)$ satisfies

$$z'(x) = \frac{y'(x)}{y(x)} = \frac{4x^2 y(x) - 2y(x)\ln y(x)}{xy(x)} = 4x - \frac{2z(x)}{x}.$$

This is a first-order linear equation with integrating factor

$$\mu(x) = \exp\left(\int \frac{2\,dx}{x}\right) = e^{2\ln x + C} = Kx^2.$$

Letting $\mu(x) = x^2$ for simplicity, we must thus have

$$\begin{aligned} \Big[\mu(x)z(x)\Big]' = 4x^3 \quad &\Longrightarrow \quad x^2 z(x) = x^4 + C \\ &\Longrightarrow \quad z(x) = x^2 + C/x^2 \quad \Longrightarrow \quad y(x) = e^{x^2 + C/x^2}. \end{aligned}$$

**2a.** The function $f(x,y) = 2x + (y-x^2)^{2/3}$ is continuous at all points and its derivative

$$\frac{\partial f}{\partial y} = \frac{2}{3} \cdot (y-x^2)^{-1/3}$$

is continuous at all points $(x,y)$ for which $y \neq x^2$. Thus, a unique solution exists as long as $y_0 \neq x_0^2$. To treat the remaining case $y_0 = x_0^2$, we note that

$$z(x) = y(x) - x^2 \quad \Longrightarrow \quad z'(x) = y'(x) - 2x = (y(x) - x^2)^{2/3} = z(x)^{2/3}.$$

In particular, $y(x)$ is a solution of the given problem if and only if $z(x)$ satisfies

$$z'(x) = z(x)^{2/3}, \qquad z(x_0) = y_0 - x_0^2 = 0.$$

We claim that the last equation does not have a unique solution. In fact, $z(x) = 0$ is obviously a solution and we may also separate variables to get

$$\begin{aligned} \frac{dz}{dx} = z^{2/3} \quad \Longrightarrow \quad \int z^{-2/3}\,dz = \int dx \quad &\Longrightarrow \quad 3z^{1/3} = x + C \\ &\Longrightarrow \quad z = \frac{(x+C)^3}{27} \end{aligned}$$

in any interval in which $z \neq 0$. This makes $z(x) = \frac{1}{27}(x-x_0)^3$ a second solution of the initial value problem, so the solution is not unique, if it happens that $y_0 = x_0^2$.

**2b.** To say that $y(x)$ satisfies the given equation is to say that $y(x)$ satisfies

$$y'(x) = 2x + \frac{y(x)}{x+1}, \qquad y(0) = 0.$$

Note that the last equation is first-order linear with integrating factor

$$\mu(x) = \exp\left(-\int \frac{dx}{x+1}\right) = e^{-\ln|x+1|+C} = K(x+1)^{-1}.$$

Letting $\mu(x) = (x+1)^{-1}$ for simplicity, we must thus have

$$\Big[\mu(x)y(x)\Big]' = \frac{2x}{x+1} = 2 - \frac{2}{x+1} \quad \Longrightarrow \quad \frac{y(x)}{x+1} = 2x - 2\ln|x+1| + C.$$

Since $y(0) = 0$, it easily follows that $C = 0$ and this finally gives

$$y(x) = 2x(x+1) - 2(x+1)\ln|x+1|.$$

**2c.** A solution of the given problem is a solution of the associated integral equation

$$y(x) = a + \int_1^x \frac{s^2+1}{s^2+s} \cdot y(s) \cdot \sin y(s)\,ds.$$

Since the sine term is at most 1, we may then use the Gronwall inequality to get

$$|y(x)| \leq a + \int_1^x \frac{s^2+1}{s^2+s} \cdot |y(s)|\, ds \quad\Longrightarrow\quad |y(x)| \leq a \exp\left(\int_1^x \frac{s^2+1}{s^2+s}\, ds\right)$$

for all $x \geq 1$. To compute the integral, we use partial fractions to write

$$\frac{s^2+1}{s^2+s} = 1 + \frac{1-s}{s^2+s}, \qquad \frac{1-s}{s^2+s} = \frac{A}{s} + \frac{B}{s+1}$$

for some constants $A, B$. Clearing denominators, one finds that

$$1 - s = A(s+1) + Bs = (A+B)s + A,$$

so it easily follows that $A = 1$ and $B = -1 - A = -2$. This actually gives

$$\begin{aligned}\int_1^x \frac{s^2+1}{s^2+s}\, ds = \int_1^x \left(1 + \frac{1}{s} - \frac{2}{s+1}\right) ds &= \Big[s + \ln s - 2\ln(s+1)\Big]_{s=1}^x \\ &= x + \ln x - 2\ln(x+1) - 1 + 2\ln 2\end{aligned}$$

for all $x \geq 1$. In view of our computations above, the solution is thus bounded by

$$|y(x)| \leq a \exp\left(\int_1^x \frac{s^2+1}{s^2+s}\, ds\right) = \frac{4axe^x}{e(x+1)^2}.$$

**3a.** To find the homogeneous solution $y_h$, we solve the associated characteristic equation

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0.$$

Noting that $\lambda = 1$ is a root, one may easily factor this polynomial to get

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda - 1) \cdot (\lambda^2 - 3\lambda + 2) = (\lambda - 1)^2 \cdot (\lambda - 2).$$

In particular, the roots are $\lambda = 1, 1, 2$ and the homogeneous solution is given by

$$y_h = c_1 e^t + c_2 te^t + c_3 e^{2t}.$$

Based on this fact, we now look for a particular solution of the form

$$y_p = Ate^{2t} + B\sin t + C\cos t.$$

Differentiating this expression three times, one can easily check that

$$y_p''' - 4y_p'' + 5y_p' - 2y_p = Ae^{2t} + (2B - 4C)\sin t + (4B + 2C)\cos t.$$

To ensure that $y_p$ is a solution of the given equation, we must thus ensure that

$$A = 1, \qquad 2B - 4C = -1, \qquad 4B + 2C = 0.$$

This gives $B = -1/10$ and also $C = -2B = 1/5$, so a particular solution is

$$y_p = te^{2t} - \frac{\sin t}{10} + \frac{\cos t}{5}.$$

We conclude that every solution of the given equation must have the form

$$y = y_h + y_p = c_1 e^t + c_2 te^t + c_3 e^{2t} + te^{2t} - \frac{\sin t}{10} + \frac{\cos t}{5}.$$

**3b.** We use the change of variables $y(t) = y_1(t)v(t) = e^{2t}v(t)$ and we note that

$$\begin{aligned} y(t) &= e^{2t}v, \\ y'(t) &= 2e^{2t}v + e^{2t}v', \\ y''(t) &= 4e^{2t}v + 4e^{2t}v' + e^{2t}v''. \end{aligned}$$

Combining these three equations, it is now easy to check that

$$ty''(t) - (4t+2)y'(t) + (4t+4)y(t) = te^{2t}v'' - 2e^{2t}v'.$$

In particular, $y(t)$ is a solution of the given equation if and only if

$$tv'' - 2v' = 0 \quad\Longleftrightarrow\quad v'' - \frac{2}{t}\,v' = 0.$$

This is really a first-order linear equation in $v'$ with integrating factor

$$\mu(t) = \exp\left(-\int \frac{2}{t}\,dt\right) = e^{-2\ln t + C} = Kt^{-2}.$$

Letting $\mu(t) = t^{-2}$ for simplicity, we must thus have

$$\begin{aligned} (\mu v')' = 0 \quad\Longrightarrow\quad v' = c_1/\mu = c_1 t^2 \quad&\Longrightarrow\quad v = c_2 t^3 + c_3 \\ &\Longrightarrow\quad y = e^{2t}v = c_2 t^3 e^{2t} + c_3 e^{2t}. \end{aligned}$$

**4a.** The given system is linear and it can be written in the form $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$, where

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}.$$

The eigenvalues of $A$ are the roots of the characteristic polynomial

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad\Longrightarrow\quad \lambda^2 - 2a\lambda + a^2 - 1 = 0 \quad\Longrightarrow\quad \lambda = a \pm 1.$$

**Case 1.** When $a < -1$, we have $a - 1 < a + 1 < 0$ and the eigenvalues are both negative. Thus, the zero solution is both stable and asymptotically stable.

**Case 2.** When $a > -1$, we have $a + 1 > 0$ and the zero solution is unstable.

**Case 3.** When $a = -1$, one of the eigenvalues is zero and the other one is negative. Thus, the zero solution is stable but not asymptotically stable.

**4b.** Let $a, b > 0$ so that the first two properties of a Lyapunov function hold. Since

$$\begin{aligned}\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x}\, x'(t) + \frac{\partial V}{\partial y}\, y'(t) &= 2ax(-x - xy^2) + 2by(-y - x^2 y) \\ &= -2ax^2 - 2ax^2y^2 - 2by^2 - 2bx^2y^2,\end{aligned}$$

we conclude that $V$ is a strict Lyapunov function for any constants $a, b > 0$.

**4c.** It is clear that $V$ satisfies the first two properties of a Lyapunov function, while

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x}\, x'(t) + \frac{\partial V}{\partial y}\, y'(t) = 2x(y - x) + 2y(x - ay).$$

Rearranging terms and completing the square, one now finds that

$$\nabla V \cdot \boldsymbol{f} = -2x^2 + 4xy - 2ay^2 = -2(x - y)^2 + 2(1 - a)y^2.$$

It easily follows that $V$ is a strict Lyapunov function if and only if $a > 1$.