MA2327, Problem set #5
(Practice problems with solutions)

1. Consider the following systems. Is the zero solution stable? Asymptotically stable?

(a) $x' = -2y, \;\; y' = 2x$ (b) $x' = x + 2y, \;\; y' = x$ (c) $x' = x - 5y, \;\; y' = x - 3y$

2. Show that $V(x, y) = 2x^2 + 2xy + y^2$ is a Lyapunov function for the system

$$x'(t) = 2x + 2y, \qquad y'(t) = -4x - 2y.$$

3. Show that $V(x, y) = \ln(1 + x^2) + y^2$ is a Lyapunov function for the system

$$x'(t) = x(y - 1), \qquad y'(t) = -\frac{x^2}{1 + x^2}.$$

4. Let $a > 0$. Show that $V(x, y) = x^2 + 2y^2$ is a strict Lyapunov function for the system

$$x'(t) = ay^2 - x, \qquad y'(t) = -y - ax^2.$$

5. For which values of a is the zero solution stable? Asymptotically stable?

$$x'(t) = ax(x^2 + y^2) - xy^3, \qquad y'(t) = ay(x^2 + y^2) + x^2y^2.$$

Hint: use the given equations to determine $E(t) = x(t)^2 + y(t)^2$ explicitly.

6. For which values of a is the zero solution stable? Asymptotically stable?

$$x'(t) = ax(t) + (a + 1)y(t), \qquad y'(t) = (a - 1)x(t) + ay(t).$$

7. Determine the critical points of the autonomous system

$$x'(t) = (x - 1)(x + y), \qquad y'(t) = y - x^2.$$

Which of these points are stable? Which of them are asymptotically stable?

8. Determine the critical points of the autonomous system

$$x'(t) = x(x + y - 1), \qquad y'(t) = x^2 - y - 1.$$

Which of these points are stable? Which of them are asymptotically stable?

1. Consider the following systems. Is the zero solution stable? Asymptotically stable?

(a) $x' = -2y,\; y' = 2x$ (b) $x' = x + 2y,\; y' = x$ (c) $x' = x - 5y,\; y' = x - 3y$

In each case, we write the given system in the form $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$ and then we compute the eigenvalues of A to determine stability. In the first case, we have

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \implies \lambda^2 + 4 = 0 \implies \lambda = \pm 2i,$$

so the zero solution is stable but not asymptotically stable. In the second case, we have

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \implies \lambda^2 - \lambda - 2 = 0 \implies \lambda = -1, 2,$$

so one eigenvalue is positive and the zero solution is unstable. In the third case, we have

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \implies \lambda^2 + 2\lambda + 2 = 0 \implies \lambda = -1 \pm i,$$

so the zero solution is both stable and asymptotically stable.

2. Show that $V(x, y) = 2x^2 + 2xy + y^2$ is a Lyapunov function for the system

$$x'(t) = 2x + 2y, \qquad y'(t) = -4x - 2y.$$

It is clear that V satisfies the first two properties of a Lyapunov function because

$$V(x, y) = 2x^2 + 2xy + y^2 = (x + y)^2 + x^2 \geq 0$$

and equality holds only at the origin. The third property also holds because

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x}\, x'(t) + \frac{\partial V}{\partial y}\, y'(t) = (4x + 2y)(2x + 2y) + (2x + 2y)(-4x - 2y) = 0.$$

3. Show that $V(x, y) = \ln(1 + x^2) + y^2$ is a Lyapunov function for the system

$$x'(t) = x(y - 1), \qquad y'(t) = -\frac{x^2}{1 + x^2}.$$

The given function is certainly continuous and we also have

$$1 + x^2 \geq 1 \implies \ln(1 + x^2) \geq 0 \implies V(x, y) = \ln(1 + x^2) + y^2 \geq 0$$

with $V(x, y) = 0$ if and only if $x = y = 0$. The result now follows by noting that

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x}\, x'(t) + \frac{\partial V}{\partial y}\, y'(t) = \frac{2x}{1 + x^2} \cdot x(y - 1) - 2y \cdot \frac{x^2}{1 + x^2} = -\frac{2x^2}{1 + x^2}.$$

4. Let $a > 0$. Show that $V(x,y) = x^2 + 2y^2$ is a strict Lyapunov function for the system

$$x'(t) = ay^2 - x, \qquad y'(t) = -y - ax^2.$$

It is clear that V satisfies the first two properties of a Lyapunov function, while

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x}\, x'(t) + \frac{\partial V}{\partial y}\, y'(t) = 2x(ay^2 - x) + 4y(-y - ax^2).$$

Rearranging terms, one may express the last equation in the form

$$\nabla V \cdot \boldsymbol{f} = -2x^2(1 + 2ay) - 2y^2(2 - ax).$$

When it comes to the open region $R = \left(-\frac{1}{a}, \frac{1}{a}\right) \times \left(-\frac{1}{4a}, \frac{1}{4a}\right)$, we thus have

$$\nabla V \cdot \boldsymbol{f} \leq -2x^2(1 - 1/2) - 2y^2(2 - 1) = -x^2 - 2y^2$$

within R. This implies that $\nabla V \cdot \boldsymbol{f} \leq 0$ and that equality holds only at the origin.

5. For which values of a is the zero solution stable? Asymptotically stable?

$$x'(t) = ax(x^2 + y^2) - xy^3, \qquad y'(t) = ay(x^2 + y^2) + x^2y^2.$$

Hint: use the given equations to determine $E(t) = x(t)^2 + y(t)^2$ explicitly.

Consider the function $E(t) = x(t)^2 + y(t)^2$. Its derivative is easily seen to satisfy

$$\begin{aligned} E'(t) &= 2x(t)x'(t) + 2y(t)y'(t) \\ &= 2ax^2(x^2 + y^2) - \cancel{2x^2y^3} + 2ay^2(x^2 + y^2) + \cancel{2x^2y^3} \\ &= 2a(x^2 + y^2)^2 = 2aE(t)^2 \end{aligned}$$

and this is a separable equation. Since $E = 0$ is a solution, it follows by uniqueness that every other solution satisfies $E \neq 0$ at all points. We may thus separate variables to get

$$\frac{dE}{dt} = 2aE^2 \quad \Longrightarrow \quad \int E^{-2}\, dE = \int 2a\, dt \quad \Longrightarrow \quad -E(t)^{-1} = 2at + C.$$

Setting $t = 0$ now gives $C = -E(0)^{-1}$ and this implies that

$$-\frac{1}{E(t)} = 2at - \frac{1}{E(0)} = \frac{2aE(0)t - 1}{E(0)} \quad \Longrightarrow \quad E(t) = \frac{E(0)}{1 - 2aE(0)t}.$$

Case 1. If it happens that $a = 0$, then $E(t)$ is constant and the distance from the origin is constant as well. Thus, the zero solution is stable but not asymptotically stable.
Case 2. If it happens that $a < 0$, then $E(t)$ is defined for all times $t \geq 0$ and it is easily seen to approach zero as $t \to \infty$. Thus, the zero solution is asymptotically stable.
Case 3. If it happens that $a > 0$, then $E(t)$ is not defined for all times $t \geq 0$. In fact, it becomes infinite when $2aE(0)t = 1$, so the zero solution is unstable.

6. For which values of a is the zero solution stable? Asymptotically stable?

$$x'(t) = ax(t) + (a+1)y(t), \qquad y'(t) = (a-1)x(t) + ay(t).$$

The given system is linear and it can be written in the form $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$, where

$$A = \begin{bmatrix} a & a+1 \\ a-1 & a \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad\Longrightarrow\quad \lambda^2 - 2a\lambda + 1 = 0 \quad\Longrightarrow\quad \lambda = a \pm \sqrt{a^2 - 1}.$$

Case 1. When $-1 < a < 1$, the eigenvalues are complex with real part equal to a. Thus, the zero solution is stable, if $-1 < a \leq 0$, and asymptotically stable, if $-1 < a < 0$.
Case 2. When $a \leq -1$, the eigenvalues are real and they are easily seen to be negative. In particular, the zero solution is both stable and asymptotically stable.
Case 3. When $a \geq 1$, the eigenvalues are real and they are easily seen to be positive. In particular, the zero solution is neither stable nor asymptotically stable.

7. Determine the critical points of the autonomous system

$$x'(t) = (x-1)(x+y), \qquad y'(t) = y - x^2.$$

Which of these points are stable? Which of them are asymptotically stable?

To find the critical points (x, y), we need to solve the system of equations

$$(x-1)(x+y) = 0, \qquad y - x^2 = 0.$$

When $x = 1$, the second equation gives $y = x^2 = 1$. When $x = -y$, it gives

$$y = x^2 = y^2 \quad\Longrightarrow\quad y = 0, 1.$$

We conclude that the critical points are $A(1,1)$, $B(0,0)$ and $C(-1,1)$. Let us now check these points for stability by computing the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2x+y-1 & x-1 \\ -2x & 1 \end{bmatrix}.$$

When it comes to the point $A(1,1)$, we have

$$J = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \quad\Longrightarrow\quad \lambda = 2, 1$$

and the eigenvalues are positive, so this point is unstable. At the point $B(0,0)$, we have

$$J = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad \lambda = -1, 1$$

and this point is unstable as well. At the point $C(-1,1)$, we have

$$J = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + \lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = \frac{-1 \pm i\sqrt{7}}{2}.$$

Thus, the eigenvalues are complex with negative real part and C is asymptotically stable.

8. Determine the critical points of the autonomous system

$$x'(t) = x(x + y - 1), \qquad y'(t) = x^2 - y - 1.$$

Which of these points are stable? Which of them are asymptotically stable?

To find the critical points (x, y), we need to solve the system of equations

$$x(x + y - 1) = 0, \qquad x^2 - y - 1 = 0.$$

When $x = 0$, the second equation gives $y = x^2 - 1 = -1$. When $x = 1 - y$, it gives

$$x^2 = y + 1 = 2 - x \quad \Longrightarrow \quad x^2 + x - 2 = 0 \quad \Longrightarrow \quad (x - 1)(x + 2) = 0.$$

We conclude that the critical points are $A(0, -1)$, $B(1, 0)$ and $C(-2, 3)$. Let us now check these points for stability by computing the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2x + y - 1 & x \\ 2x & -1 \end{bmatrix}.$$

When it comes to the point $A(0, -1)$, we have

$$J = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad \Longrightarrow \quad \lambda = -2, -1$$

so this point is both stable and asymptotically stable. At the point $B(1, 0)$, we have

$$J = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 - 3 = 0 \quad \Longrightarrow \quad \lambda = \pm\sqrt{3}$$

and one eigenvalue is positive, so this point is unstable. At the point $C(-2, 3)$, we have

$$J = \begin{bmatrix} -2 & -2 \\ -4 & -1 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + 3\lambda - 6 = 0 \quad \Longrightarrow \quad \lambda = \frac{-3 \pm \sqrt{33}}{2}.$$

Once again, one of the eigenvalues is positive, so the point C is unstable as well.