MA2327, Problem set #4
(Practice problems with solutions)

1. Solve the inhomogeneous equation $y'''(t) + 5y''(t) + 9y'(t) + 5y(t) = 2t + 3e^{2t}$.

2. Solve the inhomogeneous equation $y''(t) - y'(t) - 2y(t) = \sin t + 5e^{-t}$.

3. The function $y_1(t) = 1/t$ is easily seen to satisfy the linear equation

$$t(t+1)y''(t) + (2-t^2)y'(t) - (t+2)y(t) = 0, \qquad t > 0.$$

Use reduction of order to find a basis of solutions for this equation.

4. Show that every solution of $y''''(t) + 5y''(t) + 4y(t) = 1 + \sin(3t)$ is bounded.

5. Let $a, b \in \mathbb{R}$ be fixed. Find the unique solution of the initial value problem

$$y''(t) - 2y'(t) + 2y(t) = e^t, \qquad y(0) = a, \qquad y'(0) = b.$$

6. Let $a \in \mathbb{R}$ be fixed and consider the linear equation with non-constant coefficients

$$ty''(t) + 2y'(t) + ty(t) = at^2, \qquad t > 0.$$

Use the substitution $y(t) = t^{-1}v(t)$ in order to solve this equation explicitly.

1. Solve the inhomogeneous equation $y'''(t) + 5y''(t) + 9y'(t) + 5y(t) = 2t + 3e^{2t}$.

To find the homogeneous solution y_h, we solve the associated characteristic equation

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = 0.$$

Noting that $\lambda = -1$ is a root, one may easily factor this polynomial to get

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = (\lambda+1)\cdot(\lambda^2 + 4\lambda + 5) = (\lambda+1)\cdot((\lambda+2)^2+1).$$

In particular, the roots are $\lambda = -1$ and $\lambda = -2 \pm i$, so the homogeneous solution is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}\sin t + c_3 e^{-2t}\cos t.$$

Based on this fact, we now look for a particular solution of the form

$$y_p = At + B + Ce^{2t}.$$

Differentiating this expression three times, one finds that

$$\begin{aligned} y_p''' + 5y_p'' + 9y_p' + 5y_p &= 8Ce^{2t} + 5(4Ce^{2t}) + 9(A + 2Ce^{2t}) + 5(At + B + Ce^{2t}) \\ &= 5At + (9A + 5B) + 51Ce^{2t}. \end{aligned}$$

To ensure that y_p is a solution of the given equation, we must thus ensure that

$$5A = 2, \qquad 9A + 5B = 0, \qquad 51C = 3.$$

This gives $A = 2/5$, $B = -9A/5 = -18/25$ and also $C = 1/17$, so a particular solution is

$$y_p = \frac{2t}{5} - \frac{18}{25} + \frac{e^{2t}}{17}.$$

We conclude that every solution of the given equation must have the form

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^{-2t} \sin t + c_3 e^{-2t} \cos t + \frac{2t}{5} - \frac{18}{25} + \frac{e^{2t}}{17}.$$

2. Solve the inhomogeneous equation $y''(t) - y'(t) - 2y(t) = \sin t + 5e^{-t}$.

When it comes to the homogeneous solution y_h, one easily finds that

$$\lambda^2 - \lambda - 2 = 0 \quad \Longrightarrow \quad (\lambda - 2)(\lambda + 1) = 0 \quad \Longrightarrow \quad \lambda = 2, -1$$
$$\Longrightarrow \quad y_h = c_1 e^{2t} + c_2 e^{-t}.$$

Keeping this in mind, we shall now look for a particular solution of the form

$$y_p = A \sin t + B \cos t + Cte^{-t}.$$

Differentiating this expression twice, it is easy to check that

$$\begin{aligned} y_p' &= A \cos t - B \sin t + Ce^{-t} - Cte^{-t}, \\ y_p'' &= -A \sin t - B \cos t - 2Ce^{-t} + Cte^{-t}, \\ y_p'' - y_p' - 2y_p &= (B - 3A) \sin t - (A + 3B) \cos t - 3Ce^{-t}. \end{aligned}$$

To ensure that y_p is a solution of the given equation, we must thus ensure that

$$B - 3A = 1, \qquad A + 3B = 0, \qquad 3C = -5.$$

It easily follows that $(A, B, C) = (-3/10, 1/10, -5/3)$, so a particular solution is

$$y_p = -\frac{3 \sin t}{10} + \frac{\cos t}{10} - \frac{5te^{-t}}{3}.$$

In other words, every solution of the given equation must have the form

$$y = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - \frac{3 \sin t}{10} + \frac{\cos t}{10} - \frac{5te^{-t}}{3}.$$

3. The function $y_1(t) = 1/t$ is easily seen to satisfy the linear equation

$$t(t+1)y''(t) + (2-t^2)y'(t) - (t+2)y(t) = 0, \qquad t > 0.$$

Use reduction of order to find a basis of solutions for this equation.

We use the change of variables $y(t) = y_1(t)v(t) = t^{-1}v(t)$ and we note that

$$\begin{aligned} y(t) &= t^{-1}v, \\ y'(t) &= -t^{-2}v + t^{-1}v', \\ y''(t) &= 2t^{-3}v - 2t^{-2}v' + t^{-1}v''. \end{aligned}$$

Combining these three equations, it is easy to check that

$$t(t+1)y''(t) + (2-t^2)y'(t) - (t+2)y(t) = (t+1)v'' - (t+2)v'.$$

In particular, $y(t)$ is a solution of the given equation if and only if

$$v'' - \frac{t+2}{t+1}\, v' = 0.$$

The last equation is really a first-order linear equation in v' with integrating factor

$$\mu(t) = \exp\left(-\int \frac{t+2}{t+1}\, dt\right) = \exp\left(-\int dt - \int \frac{dt}{t+1}\right) = Ke^{-t}(t+1)^{-1}.$$

Taking $K = 1$ for simplicity, we must thus have

$$\begin{aligned} (\mu v')' = 0 \quad \Longrightarrow \quad \mu v' = c_1 \quad &\Longrightarrow \quad v' = c_1/\mu = c_1(t+1)e^t \\ &\Longrightarrow \quad v = c_1 \int (t+1)(e^t)'\, dt. \end{aligned}$$

Once we now integrate by parts, we may finally conclude that

$$v = c_1(t+1)e^t - c_1 \int e^t\, dt = c_1(t+1)e^t - c_1e^t + c_2 = c_1te^t + c_2.$$

Since $y = t^{-1}v$, every solution of the given equation has the form $y = c_1e^t + c_2t^{-1}$.

4. Show that every solution of $y''''(t) + 5y''(t) + 4y(t) = 1 + \sin(3t)$ is bounded.

To find the homogeneous solution y_h, we solve the associated characteristic equation

$$\lambda^4 + 5\lambda^2 + 4 = 0.$$

There are no real roots, as the left hand side is clearly positive whenever λ is real. On the other hand, the last equation is a quadratic equation for $\mu = \lambda^2$ and we have

$$\mu^2 + 5\mu + 4 = 0 \quad \Longrightarrow \quad (\mu + 1)(\mu + 4) = 0 \quad \Longrightarrow \quad \mu = -1, -4.$$

This means that the roots are $\lambda = \pm i$ and $\lambda = \pm 2i$, so the homogeneous solution is

$$y_h = c_1 \sin t + c_2 \cos t + c_3 \sin(2t) + c_4 \cos(2t).$$

We conclude that every solution of the inhomogeneous equation has the form

$$y = y_h + y_p = c_1 \sin t + c_2 \cos t + c_3 \sin(2t) + c_4 \cos(2t) + A + B \sin(3t) + C \cos(3t)$$

for some coefficients A, B, C. In particular, every solution is periodic and bounded.

5. Let $a, b \in \mathbb{R}$ be fixed. Find the unique solution of the initial value problem

$$y''(t) - 2y'(t) + 2y(t) = e^t, \qquad y(0) = a, \qquad y'(0) = b.$$

To find the homogeneous solution y_h, we solve the associated characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Longrightarrow \quad (\lambda - 1)^2 + 1 = 0 \quad \Longrightarrow \quad (\lambda - 1)^2 = -1.$$

This gives the complex roots $\lambda = 1 \pm i$, so the homogeneous solution is

$$y_h = c_1 e^t \sin t + c_2 e^t \cos t.$$

Based on this fact, we now look for a particular solution of the form $y_p = Ae^t$. Since

$$y_p'' - 2y_p' + 2y_p = Ae^t - 2Ae^t + 2Ae^t = Ae^t,$$

this satisfies the given equation if and only if $A = 1$. Thus, every solution has the form

$$y = y_h + y_p = c_1 e^t \sin t + c_2 e^t \cos t + e^t.$$

Next, we turn to the initial conditions and we differentiate to get

$$y'(t) = (c_1 - c_2)e^t \sin t + (c_1 + c_2)e^t \cos t + e^t.$$

To ensure that $y(0) = a$ and $y'(0) = b$, we need to ensure that

$$a = y(0) = c_2 + 1, \qquad b = y'(0) = c_1 + c_2 + 1 = c_1 + a.$$

This obviously gives $c_1 = b - a$ and $c_2 = a - 1$, so the unique solution of the problem is

$$y = (b - a)e^t \sin t + (a - 1)e^t \cos t + e^t.$$

6. Let $a \in \mathbb{R}$ be fixed and consider the linear equation with non-constant coefficients

$$ty''(t) + 2y'(t) + ty(t) = at^2, \qquad t > 0.$$

Use the substitution $y(t) = t^{-1}v(t)$ in order to solve this equation explicitly.

First, we differentiate $y(t) = t^{-1}v(t)$ twice to find that

$$y'(t) = -t^{-2}v(t) + t^{-1}v'(t), \qquad y''(t) = 2t^{-3}v(t) - 2t^{-2}v'(t) + t^{-1}v''(t).$$

In view of this computation, it is now easy to check that

$$ty'' + 2y' = v'' \quad \Longrightarrow \quad ty'' + 2y' + ty = v'' + v.$$

In particular, y is a solution of the given equation if and only if v satisfies

$$v''(t) + v(t) = at^2.$$

To find the homogeneous solution v_h, we solve the characteristic equation to get

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i \quad \Longrightarrow \quad v_h = c_1 \sin t + c_2 \cos t.$$

Next, we turn to the particular solution v_p and we seek a solution of the form

$$v_p = At^2 + Bt + C.$$

Since $v_p' = 2At + B$ and $v_p'' = 2A$, to say that v_p is a particular solution is to say that

$$at^2 = v_p'' + v_p = At^2 + Bt + (2A + C).$$

It easily follows that $(A, B, C) = (a, 0, -2a)$ and that every solution has the form

$$v(t) = v_h + v_p = c_1 \sin t + c_2 \cos t + at^2 - 2a.$$

As for the original variable $y(t) = t^{-1}v(t)$, this is obviously given by

$$y(t) = \frac{c_1 \sin t}{t} + \frac{c_2 \cos t}{t} + \frac{a(t^2 - 2)}{t}.$$