MA2327, Problem set #3
(Practice problems with solutions)

1. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$.

2. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 5 & -1 \\ 5 & 1 \end{bmatrix}$.

3. Find the unique solution of the initial value problem

$$y'''(t) - 3y''(t) + 4y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 4, \qquad y''(0) = 3.$$

4. Use the matrix exponential to solve the initial value problem

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad A = \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

5. Suppose that $\Phi(t)$ is a fundamental matrix for the system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$. Find the matrix $A(t)$ and determine the unique solution that satisfies $\boldsymbol{y}(0) = \boldsymbol{y}_0$ when

$$\Phi(t) = e^{\sin t} \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}, \qquad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6. Suppose that $\Phi(t)$ is a fundamental matrix for the system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$. Find the constant matrix A and compute its matrix exponential e^{tA} when

$$\Phi(t) = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}.$$

1. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$.

The characteristic polynomial of the given matrix is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus, $\lambda = 3$ is the only eigenvalue and it is easy to check that the only eigenvector is

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Pick any nonzero vector $\boldsymbol{v}_1$ that is not an eigenvector and let $\boldsymbol{v}_2 = (A - \lambda I)\boldsymbol{v}_1$, say

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

These vectors form a Jordan basis for A and we also have

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 3 & \\ 1 & 3 \end{bmatrix} \implies e^{tJ} = \begin{bmatrix} e^{3t} & \\ te^{3t} & e^{3t} \end{bmatrix}.$$

As for the exponential of the original matrix A, this is given by

$$e^{tA} = Be^{tJ}B^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \cdot e^{3t} \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = e^{3t} \begin{bmatrix} 1+2t & -2t \\ 2t & 1-2t \end{bmatrix}.$$

2. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 5 & -1 \\ 5 & 1 \end{bmatrix}$.

The characteristic polynomial of the given matrix is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 10 = (\lambda - 3)^2 + 1.$$

Thus, the eigenvalues are $\lambda = 3 \pm i$ and the corresponding eigenvectors turn out to be

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}.$$

This implies that A is diagonalisable and that we also have

$$B = \begin{bmatrix} 1 & 1 \\ 2-i & 2+i \end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 3+i & \\ & 3-i \end{bmatrix}$$
$$\implies e^{tJ} = \begin{bmatrix} e^{3t}e^{it} & \\ & e^{3t}e^{-it} \end{bmatrix}.$$

As for the exponential of the original matrix A, this is given by

$$\begin{aligned} e^{tA} = Be^{tJ}B^{-1} &= \begin{bmatrix} 1 & 1 \\ 2-i & 2+i \end{bmatrix} \cdot e^{3t} \begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix} \cdot \frac{1}{2i} \begin{bmatrix} 2+i & -1 \\ i-2 & 1 \end{bmatrix} \\ &= \frac{e^{3t}}{2i} \begin{bmatrix} (i+2)e^{it} + (i-2)e^{-it} & e^{-it} - e^{it} \\ 5e^{it} - 5e^{-it} & (i-2)e^{it} + (i+2)e^{-it} \end{bmatrix}. \end{aligned}$$

Using the formula $e^{\pm it} = \cos t \pm i \sin t$, we may thus conclude that

$$\begin{aligned} e^{tA} &= \frac{e^{3t}}{2i} \begin{bmatrix} 2i\cos t + 4i \sin t & -2i \sin t \\ 10i \sin t & 2i \cos t - 4i \sin t \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos t + 2\sin t & -\sin t \\ 5 \sin t & \cos t - 2\sin t \end{bmatrix}. \end{aligned}$$

3. Find the unique solution of the initial value problem

$$y'''(t) - 3y''(t) + 4y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 4, \qquad y''(0) = 3.$$

This is a homogeneous equation and the associated characteristic polynomial is

$$\lambda^3 - 3\lambda^2 + 4 = 0.$$

Noting that $\lambda = -1$ is a root, it is easy to check that

$$\lambda^3 - 3\lambda^2 + 4 = (\lambda + 1)(\lambda^2 - 4\lambda + 4) = (\lambda + 1)(\lambda - 2)^2.$$

In particular, $\lambda = 2$ is a double root and $\lambda = -1$ is a simple root, so

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t}$$

for some constants c_1, c_2, c_3. Next, we turn to the initial conditions and we note that

$$\begin{aligned} y(t) &= c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t}, \\ y'(t) &= 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} - c_3 e^{-t}, \\ y''(t) &= 4c_1 e^{2t} + 4c_2 e^{2t} + 4c_2 t e^{2t} + c_3 e^{-t}. \end{aligned}$$

This gives rise to a system of three equations in three unknowns, namely

$$1 = y(0) = c_1 + c_3, \qquad 4 = y'(0) = 2c_1 + c_2 - c_3, \qquad 3 = y''(0) = 4c_1 + 4c_2 + c_3.$$

On the other hand, row reduction of the associated augmented matrix gives

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 4 \\ 4 & 4 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus, the unique solution of the system is $(c_1, c_2, c_3) = (2, -1, -1)$ and we finally get

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t} = 2e^{2t} - t e^{2t} - e^{-t}.$$

4. Use the matrix exponential to solve the initial value problem

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad A = \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 8\lambda + 15 = (\lambda - 5)(\lambda - 3).$$

Since the eigenvalues are distinct, A is diagonalisable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = 5$ and $\lambda = 3$, respectively. This implies that

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 5 & \\ & 3 \end{bmatrix} \quad \Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{5t} & \\ & e^{3t} \end{bmatrix}.$$

In particular, the matrix exponential of the original matrix A is given by

$$\begin{aligned} e^{tA} = Be^{tJ}B^{-1} &= \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & \\ & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3e^{5t} - e^{3t} & 3e^{3t} - 3e^{5t} \\ e^{5t} - e^{3t} & 3e^{3t} - e^{5t} \end{bmatrix}. \end{aligned}$$

Next, we turn to the initial value problem. Since $\Phi(t) = e^{tA}$ is a fundamental matrix, every solution has the form $\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c} = e^{tA}\boldsymbol{c}$ for some vector $\boldsymbol{c}$ and this implies that

$$\begin{aligned} \boldsymbol{y}(0) = e^0\boldsymbol{c} = \boldsymbol{c} \quad &\Longrightarrow \quad \boldsymbol{y}(t) = e^{tA}\boldsymbol{y}(0) \\ &\Longrightarrow \quad \boldsymbol{y}(t) = \frac{1}{2} \begin{bmatrix} 3e^{5t} - e^{3t} & 3e^{3t} - 3e^{5t} \\ e^{5t} - e^{3t} & 3e^{3t} - e^{5t} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4e^{3t} - 3e^{5t} \\ 4e^{3t} - e^{5t} \end{bmatrix}. \end{aligned}$$

5. Suppose that $\Phi(t)$ is a fundamental matrix for the system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$. Find the matrix $A(t)$ and determine the unique solution that satisfies $\boldsymbol{y}(0) = \boldsymbol{y}_0$ when

$$\Phi(t) = e^{\sin t} \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}, \qquad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The fundamental matrix is itself a matrix solution of the system, so one has

$$\Phi'(t) = A(t)\Phi(t) \quad \Longrightarrow \quad A(t) = \Phi'(t) \cdot \Phi(t)^{-1}.$$

When it comes to the derivative $\Phi'(t)$, an application of the product rule gives

$$\Phi'(t) = e^{\sin t} \cos t \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} + e^{\sin t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

On the other hand, the inverse of $\Phi(t)$ is easily seen to be

$$\Phi(t)^{-1} = e^{-\sin t} \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}.$$

Once we now combine the last three equations, we find that

$$A(t) = \cos t \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix} + \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}.$$

Using this fact together with the identity $\sin^2 t + \cos^2 t = 1$, we conclude that

$$A(t) = \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos t & -1 \\ 1 & \cos t \end{bmatrix}.$$

Next, we turn to the initial value problem. Since $\Phi(t)$ is a fundamental matrix, we have

$$\begin{aligned} \boldsymbol{y}(t) = \Phi(t)\boldsymbol{c} \quad &\Longrightarrow \quad \boldsymbol{y}(0) = \Phi(0)\boldsymbol{c} \\ &\Longrightarrow \quad \boldsymbol{c} = \Phi(0)^{-1}\boldsymbol{y}(0) \\ &\Longrightarrow \quad \boldsymbol{y}(t) = \Phi(t)\Phi(0)^{-1}\boldsymbol{y}(0). \end{aligned}$$

Since $\boldsymbol{y}(0) = \boldsymbol{e}_1$ by assumption, the unique solution of the problem is thus

$$\begin{aligned} \boldsymbol{y}(t) &= e^{\sin t} \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{\sin t} \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{\sin t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \end{aligned}$$

6. Suppose that $\Phi(t)$ is a fundamental matrix for the system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$. Find the constant matrix A and compute its matrix exponential e^{tA} when

$$\Phi(t) = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}.$$

The fundamental matrix is itself a matrix solution of the system, so one has

$$\Phi'(t) = A\Phi(t) \quad \Longrightarrow \quad A = \Phi'(t) \cdot \Phi(t)^{-1}.$$

Using the given formula for $\Phi(t)$, it is thus easy to check that

$$A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

To compute the matrix exponential e^{tA}, one may now proceed in the usual way to find the eigenvalues and eigenvectors of A. However, it is also possible to compute e^{tA} by relating it to the fundamental matrix $\Phi(t)$ as follows. Consider the initial value problem

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0.$$

Since the unique solution can be expressed in the form $\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c}$, we have

$$\boldsymbol{y}(0) = \Phi(0)\boldsymbol{c} \quad \Longrightarrow \quad \boldsymbol{c} = \Phi(0)^{-1}\boldsymbol{y}(0) \quad \Longrightarrow \quad \boldsymbol{y}(t) = \Phi(t)\boldsymbol{c} = \Phi(t)\Phi(0)^{-1}\boldsymbol{y}(0).$$

On the other hand, e^{tA} is itself a fundamental matrix, so the solution is also given by

$$\boldsymbol{y}(t) = e^{tA}\widetilde{\boldsymbol{c}} \quad \Longrightarrow \quad \boldsymbol{y}(0) = \widetilde{\boldsymbol{c}} \quad \Longrightarrow \quad \boldsymbol{y}(t) = e^{tA}\widetilde{\boldsymbol{c}} = e^{tA}\boldsymbol{y}(0).$$

These two expressions must obviously coincide for any $\boldsymbol{y}(0)$ by uniqueness, hence

$$\begin{aligned} e^{tA} = \Phi(t)\Phi(0)^{-1} &= \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \end{aligned}$$