MA2327, Problem set #1
(Practice problems with solutions)

1. Solve the separable equation $(x^2+1)y'(x) = 2x^3 e^{-y(x)}$.

2. Solve the first-order linear equation $xy'(x) + y(x) = xe^{-2x}$, where $x > 0$.

3. Find the unique solution of the initial value problem

$$y'(x) - \frac{xy(x)}{x^2+1} = x, \qquad y(0) = 0.$$

4. Determine the nonzero solutions of the Bernoulli equation

$$y'(x) + 2xy(x) = 2x^3 y(x)^2.$$

5. Let $y_0 \in \mathbb{R}$ be given. Find the unique solution of the initial value problem

$$y'(x) = y(x)\,(1 - y(x)) \cdot \cos x, \qquad y(0) = y_0.$$

6. Use the substitution $z(x) = \log y(x)$ to find all positive solutions of

$$xy'(x) - y(x)\cos x + 2y(x)\log y(x) = 0, \qquad x > 0.$$

1. Solve the separable equation $(x^2+1)y'(x) = 2x^3 e^{-y(x)}$.

First of all, we separate variables to find that

$$(x^2+1)\,\frac{dy}{dx} = 2x^3 e^{-y} \quad\Longrightarrow\quad \int e^y\,dy = \int \frac{2x^3}{x^2+1}\,dx.$$

When it comes to the integral on the right hand side, division of polynomials gives

$$\frac{2x^3}{x^2+1} = 2x - \frac{2x}{x^2+1} \quad\Longrightarrow\quad \int \frac{2x^3}{x^2+1}\,dx = x^2 - \log(x^2+1) + C.$$

Once we now combine the last two equations, we may conclude that

$$e^y = x^2 - \log(x^2+1) + C \quad\Longrightarrow\quad y(x) = \log\big(x^2 - \log(x^2+1) + C\big).$$

2. Solve the first-order linear equation $xy'(x) + y(x) = xe^{-2x}$, where $x > 0$.

Since the standard form is $y'(x) + \frac{1}{x}\, y(x) = e^{-2x}$, an integrating factor is

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = e^{\log x + C} = Kx.$$

Letting $\mu(x) = x$ for simplicity, we must thus have

$$\Big[\mu(x)y(x)\Big]' = xe^{-2x} \qquad \Longrightarrow \qquad xy(x) = \int xe^{-2x}\, dx.$$

To compute the integral, one needs to integrate by parts to get

$$\begin{aligned}\int xe^{-2x}\, dx = \int x\left(-\frac{e^{-2x}}{2}\right)' dx &= -\frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}\, dx \\ &= -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C.\end{aligned}$$

Once we now combine the last two equations, we may conclude that

$$xy(x) = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad \Longrightarrow \quad y(x) = -\frac{e^{-2x}}{2} - \frac{e^{-2x}}{4x} + \frac{C}{x}.$$

3. Find the unique solution of the initial value problem

$$y'(x) - \frac{xy(x)}{x^2+1} = x, \qquad y(0) = 0.$$

The given equation is first-order linear with integrating factor

$$\mu(x) = \exp\left(-\int \frac{x\, dx}{x^2+1}\right) = \exp\left(-\frac{1}{2}\log(x^2+1) + C\right) = K(x^2+1)^{-1/2}.$$

Letting $\mu(x) = (x^2+1)^{-1/2}$ for simplicity, we must thus have

$$\Big[\mu(x)y(x)\Big]' = x(x^2+1)^{-1/2} \quad \Longrightarrow \quad \mu(x)y(x) = \int \frac{2x\, dx}{2\sqrt{x^2+1}} = \sqrt{x^2+1} + C.$$

In particular, every solution of the given equation has the form

$$\frac{y(x)}{\sqrt{x^2+1}} = \sqrt{x^2+1} + C \quad \Longrightarrow \quad y(x) = x^2 + 1 + C\sqrt{x^2+1}.$$

To ensure that $y(0) = 0$, we need to ensure that $0 = 1 + C$ and this finally gives

$$C = -1 \quad \Longrightarrow \quad y(x) = x^2 + 1 - \sqrt{x^2+1}.$$

4. Determine the nonzero solutions of the Bernoulli equation

$$y'(x) + 2xy(x) = 2x^3 y(x)^2.$$

Using the substitution $w = y^{-1}$, one easily finds that

$$w' = -y^{-2}y' = -y^{-2}(2x^3y^2 - 2xy) = -2x^3 + 2xw.$$

This is a first-order linear equation with integrating factor

$$\mu(x) = \exp\left(-\int 2x\,dx\right) = e^{-x^2+C} = Ke^{-x^2}.$$

Letting $\mu(x) = e^{-x^2}$ for simplicity, we must thus have

$$\Big[\mu(x)w(x)\Big]' = -2x^3e^{-x^2} \quad\Longrightarrow\quad e^{-x^2}w(x) = -\int 2x^3e^{-x^2}\,dx.$$

To compute the integral, one needs to integrate by parts to get

$$\begin{aligned} -\int 2x^3e^{-x^2}\,dx = \int x^2\left(e^{-x^2}\right)'\,dx &= x^2e^{-x^2} - \int 2xe^{-x^2}\,dx \\ &= x^2e^{-x^2} + e^{-x^2} + C. \end{aligned}$$

Once we now combine the last two equations, we may finally conclude that

$$\begin{aligned} e^{-x^2}w(x) = x^2e^{-x^2} + e^{-x^2} + C \quad &\Longrightarrow\quad w(x) = x^2 + 1 + Ce^{x^2} \\ &\Longrightarrow\quad y(x) = 1/(x^2 + 1 + Ce^{x^2}). \end{aligned}$$

5. Let $y_0 \in \mathbb{R}$ be given. Find the unique solution of the initial value problem

$$y'(x) = y(x)\,(1 - y(x)) \cdot \cos x, \qquad y(0) = y_0.$$

The given equation is separable and it can be written in the form

$$\frac{dy}{dx} = y(1-y)\cos x.$$

We note that each of $y = 0, 1$ are constant solutions. It follows by uniqueness that every other solution satisfies $y \neq 0, 1$ at all points. Thus, one may separate variables to get

$$\int \frac{dy}{y(1-y)} = \int \cos x\,dx = \sin x + C.$$

To compute the integral on the left hand side, we use partial fractions to write

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

Next, we clear denominators and we look at the values $y = 0, 1$ to find that

$$1 = A(1-y) + By \quad \Longrightarrow \quad A = B = 1.$$

In view of our computation above, we must thus have

$$\log|y| - \log|1-y| = \sin x + C \quad \Longrightarrow \quad \log\frac{|y|}{|1-y|} = \sin x + C.$$

Rearranging terms and setting $x = 0$, one finds that

$$\frac{y}{1-y} = Ke^{\sin x} \quad \Longrightarrow \quad \frac{y_0}{1-y_0} = K \quad \Longrightarrow \quad \frac{y}{1-y} = \frac{y_0 e^{\sin x}}{1-y_0}.$$

Solving the last equation for y, we may finally conclude that

$$\frac{1-y}{y} = \frac{1-y_0}{y_0 e^{\sin x}} \quad \Longrightarrow \quad \frac{1}{y} = \frac{1-y_0}{y_0 e^{\sin x}} + 1 \quad \Longrightarrow \quad y = \frac{y_0 e^{\sin x}}{1 - y_0 + y_0 e^{\sin x}}.$$

6. Use the substitution $z(x) = \log y(x)$ to find all positive solutions of

$$xy'(x) - y(x)\cos x + 2y(x)\log y(x) = 0, \qquad x > 0.$$

If $y(x)$ is a positive solution of the given equation, then $z(x) = \log y(x)$ satisfies

$$z'(x) = \frac{y'(x)}{y(x)} = \frac{y(x)\cos x - 2y(x)\log y(x)}{xy(x)} = \frac{\cos x}{x} - \frac{2z(x)}{x}.$$

This is a first-order linear equation with integrating factor

$$\mu(x) = \exp\left(\int \frac{2\,dx}{x}\right) = e^{2\log x + C} = Kx^2.$$

Letting $\mu(x) = x^2$ for simplicity, we must thus have

$$\Big[\mu(x)z(x)\Big]' = x\cos x \quad \Longrightarrow \quad x^2 z(x) = \int x\cos x\,dx.$$

To compute the integral, one needs to integrate by parts to get

$$\begin{aligned}\int x\cos x\,dx = \int x(\sin x)'\,dx &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C.\end{aligned}$$

Once we now combine the last two equations, we may finally conclude that

$$\begin{aligned} x^2 z(x) = x\sin x + \cos x + C \quad &\Longrightarrow \quad z(x) = x^{-1}\sin x + x^{-2}\cos x + Cx^{-2} \\ &\Longrightarrow \quad y(x) = \exp\left(x^{-1}\sin x + x^{-2}\cos x + Cx^{-2}\right).\end{aligned}$$