MA2327: Homework #4 solutions

1. Use Theorem 2.16 to prove Theorem 2.17 and then use Theorem 2.17 to solve

$$y''(t) - 2y'(t) + y(t) = e^t \log t, \qquad t > 0.$$

First, we prove Theorem 2.17. Suppose $y_1(t), y_2(t)$ are linearly independent solutions of

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0$$

and consider the corresponding inhomogeneous equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t).$$

In view of Theorem 2.16, there is a solution of the form $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$, where

$$\begin{aligned}\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t)/a(t) \end{bmatrix} \\ &= \frac{1}{W(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ f(t)/a(t) \end{bmatrix}\end{aligned}$$

and $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ denotes the Wronskian of y_1 and y_2. This gives

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \frac{1}{W(t)} \begin{bmatrix} -y_2(t)f(t)/a(t) \\ y_1(t)f(t)/a(t) \end{bmatrix},$$

so a particular solution of the inhomogeneous equation is

$$y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{a(t)W(t)}\,dt + y_2(t) \int \frac{y_1(t)f(t)}{a(t)W(t)}\,dt.$$

Next, we turn to the given equation. When it comes to the homogeneous problem, we have

$$\lambda^2 - 2\lambda + 1 = 0 \quad \Longrightarrow \quad (\lambda - 1)^2 = 0 \quad \Longrightarrow \quad y_h = c_1e^t + c_2te^t.$$

We may thus take $y_1(t) = e^t$ and $y_2(t) = te^t$. The corresponding Wronskian is

$$W(t) = e^t \cdot (te^t)' - (e^t)' \cdot te^t = e^t \cdot (e^t + te^t) - e^t \cdot te^t = e^{2t},$$

while $a(t) = 1$ and $f(t) = e^t \log t$. In particular, every solution must have the form

$$\begin{aligned} y(t) = y_h + y_p &= c_1e^t + c_2te^t - e^t \int \frac{te^t \cdot e^t \log t}{e^{2t}}\,dt + te^t \int \frac{e^t \cdot e^t \log t}{e^{2t}}\,dt \\ &= c_1e^t + c_2te^t - e^t \int t\log t\,dt + te^t \int \log t\,dt. \end{aligned}$$

To compute the last two integrals, one needs to integrate by parts. In fact, one has

$$\begin{aligned}\int t^k \log t\, dt &= \int \left(\frac{t^{k+1}}{k+1}\right)' \log t\, dt \\ &= \frac{t^{k+1}\log t}{k+1} - \int \frac{t^k}{k+1}\, dt = \frac{t^{k+1}\log t}{k+1} - \frac{t^{k+1}}{(k+1)^2} + C\end{aligned}$$

for any constant $k \neq -1$. Applying this formula twice, we may finally conclude that

$$\begin{aligned}y(t) &= c_1 e^t + c_2 t e^t - e^t \left(\frac{t^2 \log t}{2} - \frac{t^2}{4} + c_3\right) + t e^t (t \log t - t + c_4) \\ &= K_1 e^t + K_2 t e^t + \frac{t^2 e^t}{4}(2\log t - 3).\end{aligned}$$

2. Solve the inhomogeneous equation $y'''(t) - y''(t) + 3y'(t) + 5y(t) = t^2 - t + e^{2t}$.

To find the homogeneous solution y_h, we solve the associated characteristic equation

$$\lambda^3 - \lambda^2 + 3\lambda + 5 = 0.$$

Noting that $\lambda = -1$ is a root, one may easily factor this polynomial to get

$$\lambda^3 - \lambda^2 + 3\lambda + 5 = (\lambda + 1) \cdot (\lambda^2 - 2\lambda + 5) = (\lambda + 1) \cdot ((\lambda - 1)^2 + 4).$$

In particular, the roots are $\lambda = -1$ and $\lambda = 1 \pm 2i$, so the homogeneous solution is

$$y_h = c_1 e^{-t} + c_2 e^t \sin(2t) + c_3 e^t \cos(2t).$$

Based on this fact, we now look for a particular solution of the form

$$y_p = At^2 + Bt + C + De^{2t}.$$

Differentiating this expression three times, one finds that

$$y_p' = 2At + B + 2De^{2t}, \qquad y_p'' = 2A + 4De^{2t}, \qquad y_p''' = 8De^{2t}$$

and we may thus combine these equations to conclude that

$$y_p''' - y_p'' + 3y_p' + 5y_p = (3B - 2A + 5C) + (6A + 5B)t + 5At^2 + 15De^{2t}.$$

To ensure that y_p is a solution of the given equation, we need to ensure that

$$15D = 5A = 1, \qquad 6A + 5B = -1, \qquad 3B - 2A + 5C = 0.$$

It easily follows that $(A, B, C, D) = (1/5, -11/25, 43/125, 1/15)$, so a particular solution is

$$y_p = \frac{t^2}{5} - \frac{11t}{25} + \frac{43}{125} + \frac{e^{2t}}{15}.$$

We conclude that every solution of the given equation must have the form

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^t \sin(2t) + c_3 e^t \cos(2t) + \frac{t^2}{5} - \frac{11t}{25} + \frac{43}{125} + \frac{e^{2t}}{15}.$$

3. Solve the inhomogeneous equation $y''(t) - 3y'(t) + 2y(t) = 2\cos t - 3t + 4e^{2t}$.

When it comes to the homogeneous solution y_h, one easily finds that

$$\begin{aligned} \lambda^2 - 3\lambda + 2 = 0 \quad &\Longrightarrow \quad (\lambda-1)(\lambda-2) = 0 \quad \Longrightarrow \quad \lambda = 1, 2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \Longrightarrow \quad y_h = c_1 e^t + c_2 e^{2t}. \end{aligned}$$

Keeping this in mind, we shall now look for a particular solution of the form

$$y_p = A\sin t + B\cos t + Ct + D + Ete^{2t}.$$

Differentiating this expression twice, it is easy to check that

$$\begin{aligned} y_p' &= A\cos t - B\sin t + C + Ee^{2t} + 2Ete^{2t}, \\ y_p'' &= -A\sin t - B\cos t + 4Ee^{2t} + 4Ete^{2t}, \\ y_p'' - 3y_p' + 2y_p &= (A+3B)\sin t + (B-3A)\cos t + (2D-3C) + 2Ct + Ee^{2t}. \end{aligned}$$

To ensure that y_p is a solution of the given equation, we must thus ensure that

$$A + 3B = 0, \qquad B - 3A = 2, \qquad 2D - 3C = 0, \qquad 2C = -3, \qquad E = 4.$$

It easily follows that $(A, B, C, D, E) = (-3/5, 1/5, -3/2, -9/4, 4)$, so a particular solution is

$$y_p = -\frac{3\sin t}{5} + \frac{\cos t}{5} - \frac{3t}{2} - \frac{9}{4} + 4te^{2t}.$$

In other words, every solution of the given equation must have the form

$$y = y_h + y_p = c_1 e^t + c_2 e^{2t} - \frac{3\sin t}{5} + \frac{\cos t}{5} - \frac{3t}{2} - \frac{9}{4} + 4te^{2t}.$$

4. The function $y_1(t) = t$ is easily seen to satisfy the linear equation

$$t^3 y''(t) - ty'(t) + y(t) = 0, \qquad t > 0.$$

Use reduction of order to find a basis of solutions for this equation.

We use the change of variables $y(t) = y_1(t)v(t) = tv(t)$ and we note that

$$\begin{aligned} y'(t) &= v(t) + tv'(t), \\ y''(t) &= 2v'(t) + tv''(t), \\ t^3 y''(t) - ty'(t) + y(t) &= t^4 v''(t) + (2t^3 - t^2)v'(t). \end{aligned}$$

This means that $y(t)$ is a solution of the given equation if and only if

$$v''(t) + \left(\frac{2}{t} - \frac{1}{t^2}\right) v'(t) = 0.$$

The last equation is really a first-order linear equation in $v'(t)$ with integrating factor

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt - \int \frac{1}{t^2}\,dt\right) = e^{2\log t + 1/t + C} = Kt^2 e^{1/t}.$$

Taking $K = 1$ for simplicity, we must thus have

$$\begin{aligned} (\mu v')' = 0 \quad &\Longrightarrow \quad \mu v' = c_1 \quad \Longrightarrow \quad v' = c_1/\mu = c_1 t^{-2} e^{-1/t} \\ &\qquad\qquad\qquad\quad\;\; \Longrightarrow \quad v = c_1 \int t^{-2} e^{-1/t}\,dt = c_1 \int e^u\,du, \end{aligned}$$

where $u = -1/t$. It easily follows that every solution has the form

$$v = c_1 e^u + c_2 = c_1 e^{-1/t} + c_2 \quad \Longrightarrow \quad y(t) = tv(t) = c_1 t e^{-1/t} + c_2 t.$$