MA2327: Homework #3 solutions

1. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$.

The characteristic polynomial of the given matrix is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Thus, $\lambda = 4$ is the only eigenvalue and it is easy to check that the only eigenvector is

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Pick any nonzero vector $\boldsymbol{v}_1$ that is not an eigenvector and let $\boldsymbol{v}_2 = (A - \lambda I)\boldsymbol{v}_1$, say

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 4I)\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

These vectors form a Jordan basis for A and we also have

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 4 & \\ 1 & 4 \end{bmatrix} \quad \Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{4t} & \\ te^{4t} & e^{4t} \end{bmatrix}.$$

As for the exponential of the original matrix A, this is given by

$$e^{tA} = Be^{tJ}B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} \cdot e^{4t} \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1/2 \\ 0 & 1/4 \end{bmatrix} = e^{4t} \begin{bmatrix} 1 - 2t & -t \\ 4t & 1 + 2t \end{bmatrix}.$$

2. Compute the matrix exponential e^{tA} in the case that $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

The characteristic polynomial of the given matrix is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1,$$

so its eigenvalues are $\lambda = 2 \pm i$. The corresponding eigenvectors turn out to be

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i - 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i - 1 \end{bmatrix}.$$

This implies that A is diagonalisable and that we also have

$$B = \begin{bmatrix} 1 & 1 \\ i - 1 & -i - 1 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 2 + i & \\ & 2 - i \end{bmatrix}$$

$$\Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{2t}e^{it} & \\ & e^{2t}e^{-it} \end{bmatrix}.$$

As for the exponential of the original matrix A, this is given by

$$e^{tA} = Be^{tJ}B^{-1} = \begin{bmatrix} 1 & 1 \\ i-1 & -i-1 \end{bmatrix} \cdot e^{2t} \begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix} \cdot \frac{1}{-2i} \begin{bmatrix} -i-1 & -1 \\ 1-i & 1 \end{bmatrix}$$
$$= -\frac{e^{2t}}{2i} \begin{bmatrix} -(1+i)e^{it} + (1-i)e^{-it} & e^{-it} - e^{it} \\ 2e^{it} - 2e^{-it} & (1-i)e^{it} - (1+i)e^{-it} \end{bmatrix}.$$

Using the formula $e^{\pm it} = \cos t \pm i \sin t$, we may thus conclude that

$$e^{tA} = -\frac{e^{2t}}{2i} \begin{bmatrix} -2i\cos t - 2i\sin t & -2i\sin t \\ 4i\sin t & 2i\sin t - 2i\cos t \end{bmatrix}$$
$$= e^{2t} \begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}.$$

3. Find the unique solution of the initial value problem

$$y'''(t) - 4y''(t) - 3y'(t) + 18y(t) = 0, \qquad y(0) = y'(0) = 0, \qquad y''(0) = 25.$$

The associated characteristic equation is given by

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0.$$

Noting that $\lambda = 3$ is a root, it is easy to check that

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = (\lambda - 3)(\lambda^2 - \lambda - 6) = (\lambda - 3)(\lambda + 2)(\lambda - 3).$$

In particular, $\lambda = 3$ is a double root and $\lambda = -2$ is a simple root, so

$$y = c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{-2t}$$

for some constants c_1, c_2, c_3. Next, we turn to the initial conditions and we note that

$$y(t) = c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{-2t},$$
$$y'(t) = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} - 2c_3 e^{-2t},$$
$$y''(t) = 9c_1 e^{3t} + 6c_2 e^{3t} + 9c_2 t e^{3t} + 4c_3 e^{-2t}.$$

This gives rise to a system of three equations in three unknowns, namely

$$0 = y(0) = c_1 + c_3, \qquad 0 = y'(0) = 3c_1 + c_2 - 2c_3, \qquad 25 = y''(0) = 9c_1 + 6c_2 + 4c_3.$$

On the other hand, row reduction of the associated augmented matrix gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 \\ 9 & 6 & 4 & 25 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Thus, the unique solution of the system is $(c_1, c_2, c_3) = (-1, 5, 1)$ and we finally get

$$y = c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{-2t} = e^{-2t} - e^{3t} + 5te^{3t}.$$

4. The method of integrating factors can also be used to solve linear systems such as

$$\boldsymbol{y}'(t) + f'(t)\boldsymbol{y}(t) = A\boldsymbol{y}(t), \qquad A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

Solve this system explicitly by letting $\boldsymbol{z}(t) = e^{f(t)}\boldsymbol{y}(t)$. Hint: show that $\boldsymbol{z}'(t) = A\boldsymbol{z}(t)$.

The auxiliary variable $\boldsymbol{z}(t) = e^{f(t)}\boldsymbol{y}(t)$ is easily seen to satisfy

$$\boldsymbol{z}'(t) = f'(t)e^{f(t)}\boldsymbol{y}(t) + e^{f(t)}\boldsymbol{y}'(t) = e^{f(t)} \cdot A\boldsymbol{y}(t) = A\boldsymbol{z}(t).$$

It thus suffices to solve this linear system. The characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2),$$

so the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$. The corresponding eigenvectors turn out to be

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

This means that the solution of the system $\boldsymbol{z}'(t) = A\boldsymbol{z}(t)$ can be expressed in the form

$$\boldsymbol{z}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 = \begin{bmatrix} c_1 e^{5t} - 4c_2 e^{-2t} \\ c_1 e^{5t} + 3c_2 e^{-2t} \end{bmatrix}.$$

As for the solution of the original system, this is obviously given by

$$\boldsymbol{y}(t) = e^{-f(t)}\boldsymbol{z}(t) = e^{-f(t)} \begin{bmatrix} c_1 e^{5t} - 4c_2 e^{-2t} \\ c_1 e^{5t} + 3c_2 e^{-2t} \end{bmatrix}.$$