MA2327: Homework #2 solutions

1. Let $x_0, y_0 \in \mathbb{R}$ be given and consider the initial value problem

$$y'(x) = \frac{xy(x)}{x-1}, \qquad y(x_0) = y_0.$$

For which values of x_0, y_0 is the solution unique? What can you say for the remaining values? Hint: A similar problem appears in the notes online; see Chapter 1, page 30.

I should have perhaps stated this problem in the alternative form

$$(x-1)y'(x) = xy(x), \qquad y(x_0) = y_0$$

because the other form is not defined when $x = 1$. Since both $f = \frac{xy}{x-1}$ and $\frac{\partial f}{\partial y} = \frac{x}{x-1}$ are continuous whenever $x \neq 1$, a unique solution exists whenever $x_0 \neq 1$.

Suppose now that $x_0 = 1$. Then the given equation implies that

$$xy(x) = (x-1)y'(x) \quad \Longrightarrow \quad y(1) = 0 \quad \Longrightarrow \quad y_0 = y(x_0) = 0,$$

so there are no solutions when $y_0 \neq 0$. To treat the remaining case $y_0 = 0$, we note that

$$(x-1) \cdot \frac{dy}{dx} = xy \quad \Longrightarrow \quad \int \frac{dy}{y} = \int \frac{x}{x-1}\,dx = \int \left(1 + \frac{1}{x-1}\right) dx$$

at all points at which $x \neq 1$ and $y \neq 0$. This provides the explicit formula

$$\log|y| = x + \log|x-1| + C \quad \Longrightarrow \quad y = K(x-1)e^x$$

at all points at which $x \neq 1$ and $y \neq 0$. On the other hand, it is easy to check that

$$\begin{aligned} y = K(x-1)e^x \quad &\Longrightarrow \quad y' = Ke^x + K(x-1)e^x = Kxe^x \\ &\Longrightarrow \quad (x-1)y' = Kx(x-1)e^x = xy. \end{aligned}$$

In other words, $y = K(x-1)e^x$ satisfies the given equation at all points. Since $y(1) = 0$ and the constant K is arbitrary, there are infinitely many solutions when $x_0 = 1$ and $y_0 = 0$.

2. Let $a > 0$ be given and consider the initial value problem

$$y'(x) = \frac{x^2+3}{x^3+x} \cdot y(x) \cdot \sin y(x), \qquad y(1) = a.$$

Use the associated integral equation to show that $|y(x)| \leq \frac{2ax^3}{x^2+1}$ for all $x \geq 1$. Hint: the sine term is at most 1; use the Gronwall inequality and then partial fractions.

A solution of the given problem is a solution of the associated integral equation

$$y(x) = a + \int_1^x \frac{s^2+3}{s^3+s} \cdot y(s) \cdot \sin y(s)\, ds.$$

Since the sine term is at most 1, we may then use the Gronwall inequality to get

$$|y(x)| \leq a + \int_1^x \frac{s^2+3}{s^3+s} \cdot |y(s)|\, ds \quad \Longrightarrow \quad |y(x)| \leq a \exp\left(\int_1^x \frac{s^2+3}{s^3+s}\, ds \right)$$

for all $x \geq 1$. To compute the integral, we use partial fractions to write

$$\frac{s^2+3}{s^3+s} = \frac{s^2+3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

for some constants A, B, C. Clearing denominators, we now find that

$$s^2 + 3 = A(s^2+1) + (Bs+C)s = (A+B)s^2 + Cs + A,$$

so one may compare coefficients to get $A + B = 1$, $C = 0$ and $A = 3$. This gives

$$\begin{aligned} \int_1^x \frac{s^2+3}{s^3+s}\, ds = \int_1^x \left(\frac{3}{s} - \frac{2s}{s^2+1} \right) ds &= \Big[3\log s - \log(s^2+1) \Big]_{s=1}^x \\ &= 3\log x - \log(x^2+1) + \log 2 \end{aligned}$$

for all $x \geq 1$. In view of our computations above, the solution is thus bounded by

$$|y(x)| \leq a \exp\left(\int_1^x \frac{s^2+3}{s^3+s}\, ds \right) = \frac{2ax^3}{x^2+1}.$$

3. Find a basis of solutions for the linear homogeneous system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t), \qquad A(t) = \begin{bmatrix} 2t/(t^2+1) & 0 \\ 2t & 2t \end{bmatrix}.$$

The given matrix is lower triangular and it corresponds to the system of equations

$$x'(t) = \frac{2tx(t)}{t^2+1}, \qquad y'(t) = 2tx(t) + 2ty(t).$$

These equations are both first-order linear. The leftmost one has integrating factor

$$\mu(t) = \exp\left(-\int \frac{2t\, dt}{t^2+1} \right) = e^{-\log(t^2+1)+C} = \frac{K}{t^2+1}$$

and we may certainly take $K = 1$ for simplicity. In particular, we have

$$\Big[\mu(t)x(t)\Big]' = 0 \quad \Longrightarrow \quad \mu(t)x(t) = c_1 \quad \Longrightarrow \quad x(t) = \frac{c_1}{\mu(t)} = c_1(t^2+1).$$

Let us now turn to the equation that involves $y(t)$ and write

$$y'(t) - 2ty(t) = 2tx(t) = 2c_1 t(t^2+1).$$

In this case, an integrating factor is given by $\mu(t) = e^{-t^2}$ and one has

$$\Big[e^{-t^2}y(t)\Big]' = 2c_1 t(t^2+1)e^{-t^2} \quad \Longrightarrow \quad e^{-t^2}y(t) = \int 2c_1 t(t^2+1)e^{-t^2}\,dt.$$

To compute the integral, we use the substitution $w = t^2$ which gives $dw = 2t\,dt$ and

$$\begin{aligned} e^{-t^2}y(t) &= \int c_1(w+1)e^{-w}\,dw = \int c_1(w+1)\Big(-e^{-w}\Big)'\,dw \\ &= -c_1(w+1)e^{-w} + \int c_1 e^{-w}\,dw \\ &= -c_1(w+1)e^{-w} - c_1e^{-w} + c_2. \end{aligned}$$

Since $w = t^2$ by above, this actually shows that

$$e^{-t^2}y(t) = -c_1(w+2)e^{-w} + c_2 \quad \Longrightarrow \quad y(t) = -c_1(t^2+2) + c_2e^{t^2}.$$

In particular, every solution of the given system must have the form

$$\boldsymbol{y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} t^2+1 \\ -(t^2+2) \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{t^2} \end{bmatrix}.$$

4. Use the eigenvector method to solve the linear homogeneous system

$$\left\{ \begin{aligned} x'(t) &= 5x(t) + 2y(t) - 4z(t) \\ y'(t) &= 4x(t) + 7y(t) - 8z(t) \\ z'(t) &= 4x(t) + 2y(t) - 3z(t) \end{aligned} \right\}.$$

We need to solve the system $\boldsymbol{y}' = A\boldsymbol{y}$ in the case that A is the 3×3 matrix

$$A = \begin{bmatrix} 5 & 2 & -4 \\ 4 & 7 & -8 \\ 4 & 2 & -3 \end{bmatrix}.$$

As one can easily check, the characteristic polynomial of this matrix is given by

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 23\lambda + 15.$$

Noting that $\lambda = 1$ is a root, one may now factor this polynomial to get

$$f(\lambda) = -(\lambda - 1)(\lambda^2 - 8\lambda + 15) = -(\lambda - 1)(\lambda - 3)(\lambda - 5).$$

This means that the eigenvalues are $\lambda = 1, 3, 5$. The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

In particular, A is diagonalisable and every solution of the system has the form

$$\boldsymbol{y} = c_1 e^t \boldsymbol{v}_1 + c_2 e^{3t} \boldsymbol{v}_2 + c_3 e^{5t} \boldsymbol{v}_3 = \begin{bmatrix} c_1 e^t + c_2 e^{3t} + c_3 e^{5t} \\ 2c_1 e^t + c_2 e^{3t} + 2c_3 e^{5t} \\ 2c_1 e^t + c_2 e^{3t} + c_3 e^{5t} \end{bmatrix}.$$