MA2327: Homework #1 solutions

1. Solve the separable equation $y'(x) = xe^{y(x)-x}$.

First of all, we separate variables to find that

$$\frac{dy}{dx} = xe^y e^{-x} \quad \Longrightarrow \quad \int e^{-y}\, dy = \int xe^{-x}\, dx.$$

When it comes to the integral on the right hand side, an integration by parts gives

$$\int xe^{-x}\, dx = -\int x(e^{-x})' dx = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + C.$$

Once we now combine the last two equations, we easily find that

$$\begin{aligned} -e^{-y} = -xe^{-x} - e^{-x} + C \quad &\Longrightarrow \quad e^{-y} = xe^{-x} + e^{-x} - C \\ &\Longrightarrow \quad y = -\log(xe^{-x} + e^{-x} - C). \end{aligned}$$

2. Suppose that $y_0 > 0$ and consider the initial value problem

$$y'(x) = y(x)^3 - y(x), \qquad y(0) = y_0.$$

For which values of the constant $y_0 > 0$ is the corresponding solution global?

The given equation is separable and it can be written in the form

$$\frac{dy}{dx} = y^3 - y = y(y^2 - 1) = y(y-1)(y+1).$$

We note that each of $y = \pm 1, 0$ are constant solutions. It follows by uniqueness that every other solution satisfies $y \neq \pm 1, 0$ at all points. Thus, one may separate variables to get

$$\int \frac{dy}{y(y-1)(y+1)} = \int dx.$$

To compute the integral on the left hand side, we use partial fractions to write

$$\frac{1}{y(y-1)(y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}.$$

Next, we clear denominators and we look at the values $y = \pm 1, 0$ to find that

$$1 = A(y-1)(y+1) + By(y+1) + Cy(y-1) \quad \Longrightarrow \quad B = C = 1/2, \quad A = -1.$$

In view of our computation above, we must thus have

$$-\log|y| + \frac{1}{2}\log|y-1| + \frac{1}{2}\log|y+1| = x + C \quad\Longrightarrow\quad \log\frac{|y^2-1|}{y^2} = 2x + 2C.$$

Rearranging terms and simplifying, we conclude that

$$\frac{y^2-1}{y^2} = Ke^{2x} \quad\Longrightarrow\quad 1 - \frac{1}{y^2} = Ke^{2x} \quad\Longrightarrow\quad 1 - \frac{1}{y_0^2} = K. \tag{$*$}$$

First approach. Since $1 - Ke^{2x} = 1/y^2$ must be positive, the solution is only defined when

$$1 - Ke^{2x} > 0.$$

If it happens that $K \leq 0$, then the last equation holds for all x and the solution is global. If it happens that $K > 0$, then the solution is only defined when

$$Ke^{2x} < 1 \quad\Longleftrightarrow\quad e^{2x} < K^{-1} \quad\Longleftrightarrow\quad 2x < \log K^{-1}.$$

Since y_0 is positive by assumption, the solution is thus global if and only if

$$K \leq 0 \quad\Longleftrightarrow\quad 1 - \frac{1}{y_0^2} \leq 0 \quad\Longleftrightarrow\quad y_0^2 \leq 1 \quad\Longleftrightarrow\quad 0 < y_0 \leq 1.$$

Second approach. One may reach the same conclusion by finding the solution explicitly, but this approach is somewhat messier. More precisely, one may use equation $(*)$ to find that

$$\frac{1}{y^2} = 1 - Ke^{2x} \quad\Longrightarrow\quad y^2 = \frac{1}{1-Ke^{2x}} = \frac{y_0^2}{y_0^2 - (y_0^2-1)e^{2x}}$$
$$\Longrightarrow\quad y = \frac{y_0}{\sqrt{y_0^2 + (1-y_0^2)e^{2x}}}.$$

If it happens that $1 - y_0^2 \geq 0$, then the square root is defined for all x and the solution is global. If it happens that $1 - y_0^2 < 0$, then the solution is only defined when

$$y_0^2 + (1-y_0^2)e^{2x} > 0 \quad\Longleftrightarrow\quad (y_0^2-1)e^{2x} < y_0^2 \quad\Longleftrightarrow\quad 2x < \log\frac{y_0^2}{y_0^2-1}.$$

In particular, one needs to have $1 - y_0^2 \geq 0$ and this gives $0 < y_0 \leq 1$ as before.

3. Solve the first-order linear equation $xy'(x) - y(x) = x^3 \sin x$, where $x > 0$.

The standard form is $y'(x) - \frac{1}{x}\,y(x) = x^2 \sin x$, so an integrating factor is

$$\mu(x) = \exp\left(-\int\frac{dx}{x}\right) = e^{-\log x + C} = Kx^{-1}.$$

Letting $\mu(x) = x^{-1}$ for simplicity, we must then have

$$\Big[x^{-1}y\Big]' = x\sin x \quad \Longrightarrow \quad \frac{y}{x} = \int x\sin x\,dx.$$

To compute the integral, one needs to integrate by parts to get

$$\begin{aligned}\int x\sin x\,dx &= -\int x(\cos x)'\,dx \\ &= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.\end{aligned}$$

Thus, one may simply combine the last two equations to conclude that

$$\frac{y}{x} = -x\cos x + \sin x + C \quad \Longrightarrow \quad y = -x^2\cos x + x\sin x + Cx.$$

4. Determine the nonzero solutions of the Bernoulli equation

$$y'(x) + y(x) = xe^{2x}y(x)^3, \qquad y(0) = y_0.$$

On which interval is the solution defined in the case that y_0 is nonzero?

Using the substitution $w = y^{-2}$, one easily finds that

$$w' = -2y^{-3}y' = -2y^{-3}(xe^{2x}y^3 - y) = -2xe^{2x} + 2w.$$

This is a first-order linear equation with integrating factor

$$\mu(x) = \exp\Big(-\int 2\,dx\Big) = e^{-2x+C} = Ke^{-2x}.$$

Let us take $\mu(x) = e^{-2x}$ for simplicity. We must then have

$$\begin{aligned} w' - 2w = -2xe^{2x} \quad &\Longrightarrow \quad (e^{-2x}w)' = -2x \\ &\Longrightarrow \quad e^{-2x}w = -x^2 + C \quad \Longrightarrow \quad w = (C - x^2)e^{2x}.\end{aligned}$$

Setting $x = 0$ gives $C = w(0) = 1/y_0^2$ and this implies that

$$C - x^2 = we^{-2x} = \frac{e^{-2x}}{y^2} > 0 \quad \Longrightarrow \quad x^2 < C = \frac{1}{y_0^2}.$$

In particular, the solution is defined in the interval $(-1/|y_0|, 1/|y_0|)$. You were not asked to find the solution explicitly, but one may use the computations above to get

$$y^2 = \frac{1}{w} = \frac{e^{-2x}}{C - x^2} = \frac{e^{-2x}}{1/y_0^2 - x^2} = \frac{y_0^2e^{-2x}}{1 - y_0^2x^2} \quad \Longrightarrow \quad y = \pm\frac{y_0e^{-x}}{\sqrt{1 - y_0^2x^2}}.$$