Chapter 3. Stability theory Lecture notes for MA2327

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Definition 3.1 – Autonomous systems

A system of ordinary differential equations is called autonomous, if it has the form y'(t) = f(y(t)) for some vector-valued function f.

 \bullet We shall mainly focus on the 2×2 case and study the system

$$\left\{ \begin{array}{l} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{array} \right\}.$$

- Suppose that the functions f, g are continuously differentiable. Then there is a unique solution for each initial condition (x_0, y_0) and each solution (x(t), y(t)) corresponds to a differentiable curve in \mathbb{R}^2 .
- A phase portrait for a given system is a graphical depiction of all the curves that correspond to solutions of the system. Every point must lie on some curve and distinct curves do not intersect by uniqueness.

• First of all, we shall analyse the phase portrait of the linear system

 $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$

when A is a 2×2 diagonalisable matrix with eigenvalues $\lambda_1, \lambda_2 \neq 0$. • In this case, $J = B^{-1}AB$ is diagonal for some matrix B, so one may use the change of variables $\boldsymbol{z}(t) = B^{-1}\boldsymbol{y}(t)$ to find that

$$\mathbf{z}'(t) = B^{-1}\mathbf{y}'(t) = B^{-1}A\mathbf{y}(t) = B^{-1}AB\mathbf{z}(t) = J\mathbf{z}(t).$$

This is a diagonal system that we can readily solve to get

$$z'_k(t) = \lambda_k z_k(t) \implies z_k(t) = c_k e^{\lambda_k t}.$$

• In particular, the new variables z_1, z_2 satisfy a relation of the form

$$z_2 = c_2 e^{\lambda_2 t} = c_2 \left(\frac{c_1 e^{\lambda_1 t}}{c_1}\right)^{\lambda_2/\lambda_1} = K z_1^{\lambda_2/\lambda_1}, \quad \text{if } c_1 \neq 0.$$

• In terms of the new variable z(t), the solution of the system is

$$\boldsymbol{z}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} = c_1 e^{\lambda_1 t} \boldsymbol{e}_1 + c_2 e^{\lambda_2 t} \boldsymbol{e}_2.$$

• In terms of the original variable $\boldsymbol{y}(t)$, the solution is thus

$$\boldsymbol{y}(t) = B\boldsymbol{z}(t) = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_2 t} \boldsymbol{v}_2,$$

where v_k is an eigenvector of A with eigenvalue λ_k for each k.

- When c₁ = 0, the last equation describes the line that contains the vector v₂. When c₂ = 0, it describes the line that contains v₁.
- If it happens that λ₁ < λ₂ < 0, then every solution curve approaches the origin as t → +∞ and its direction is parallel to v₂ as t → +∞.
- If it happens that $\lambda_1 < 0 < \lambda_2$, then every solution curve must be parallel to v_1 as $t \to -\infty$ and also parallel to v_2 as $t \to +\infty$.

• Next, we analyse the phase portrait of the 2×2 linear system

 $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$

when A is a non-diagonalisable matrix with a single eigenvalue $\lambda \neq 0$. • Once again, we let $J = B^{-1}AB$ be the Jordan form of A and we use the change of variables $\boldsymbol{z}(t) = B^{-1}\boldsymbol{y}(t)$ to find that

$$z'(t) = B^{-1}y'(t) = B^{-1}Ay(t) = B^{-1}ABz(t) = Jz(t)$$

• Since e^{tJ} is a fundamental matrix for this system, we conclude that

$$\boldsymbol{z}(t) = e^{tJ}\boldsymbol{c} = \begin{bmatrix} e^{\lambda t} \\ te^{\lambda t} & e^{\lambda t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda t} \\ (c_1 t + c_2) e^{\lambda t} \end{bmatrix}$$

• In particular, the new variables z_1, z_2 satisfy a relation of the form

$$\frac{z_2}{z_1} = \frac{c_1 t + c_2}{c_1} = t + \frac{c_2}{c_1}, \quad \text{if } c_1 \neq 0.$$

• In terms of the new variable z(t), the solution of the system is

$$\boldsymbol{z}(t) = c_1 e^{\lambda t} \boldsymbol{e}_1 + (c_1 t + c_2) e^{\lambda t} \boldsymbol{e}_2.$$

• In terms of the original variable $\boldsymbol{y}(t)$, the solution is thus

$$\boldsymbol{y}(t) = B\boldsymbol{z}(t) = c_1 e^{\lambda t} \boldsymbol{v}_1 + (c_1 t + c_2) e^{\lambda t} \boldsymbol{v}_2,$$

where v_1, v_2 are the columns of B which form a Jordan basis for A.

- When $c_1 = 0$, the last equation reduces to $y(t) = c_2 e^{\lambda t} v_2$ and this describes the line which contains the eigenvector v_2 .
- If it happens that λ < 0, then every solution must approach the origin as t → +∞ and its direction must be parallel to v₂ as t → +∞.
- If it happens that λ > 0, then every solution must approach the origin as t → -∞ and its direction must be parallel to v₂ as t → -∞.

• Finally, we analyse the phase portrait of the 2×2 linear system

 $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$

when the eigenvalues of A are $\lambda=a\pm ib,$ where $a,b\in\mathbb{R}$ and b>0.

• Since the eigenvalues are distinct, the corresponding eigenvectors are linearly independent and they are also complex conjugates because

$$A\boldsymbol{v} = (a+ib)\boldsymbol{v} \implies A\overline{\boldsymbol{v}} = (a-ib)\overline{\boldsymbol{v}}.$$

• Let us write $v = v_1 + iv_2$ for some real vectors v_1, v_2 . To see that these vectors must be linearly independent, we note that

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 = 0 \implies \frac{c_1}{2} \left(\boldsymbol{v} + \overline{\boldsymbol{v}} \right) + \frac{c_2}{2i} \left(\boldsymbol{v} - \overline{\boldsymbol{v}} \right) = 0$$
$$\implies ic_1 + c_2 = ic_1 - c_2 = 0$$
$$\implies c_1 = c_2 = 0.$$

• We now use the vectors $oldsymbol{v}_1, oldsymbol{v}_2$ to obtain a basis of real solutions.

• Since $v = v_1 + iv_2$ is an eigenvector with eigenvalue $\lambda = a + ib$,

$$\boldsymbol{y}(t) = e^{\lambda t} \boldsymbol{v} = e^{at} (\cos(bt) + i\sin(bt)) \cdot (\boldsymbol{v}_1 + i\boldsymbol{v}_2)$$

is a solution of the system. The same is true for its conjugate and for any linear combination of the two, so both the real and the imaginary part of y(t) are solutions. Let us then consider the functions

- These are real solutions of the system which are linearly independent when t = 0, so they actually form a basis for the space of solutions.
- Thus, every solution of the system can be expressed in the form

$$\boldsymbol{y}(t) = c_1 \boldsymbol{y}_1(t) + c_2 \boldsymbol{y}_2(t).$$

Let B be the matrix whose columns are the vectors v₁, v₂. Then B is invertible and the change of variables z(t) = B⁻¹y(t) gives

$$\mathbf{z}(t) = c_1 e^{at} \begin{bmatrix} \cos(bt) \\ -\sin(bt) \end{bmatrix} + c_2 e^{at} \begin{bmatrix} \sin(bt) \\ \cos(bt) \end{bmatrix}$$

• In particular, the new variables z_1, z_2 satisfy a relation of the form

$$z_1(t)^2 + z_2(t)^2 = e^{2at} (c_1 \cos(bt) + c_2 \sin(bt))^2 + e^{2at} (-c_1 \sin(bt) + c_2 \cos(bt))^2 = e^{2at} (c_1^2 + c_2^2).$$

If it happens that a = 0, then the last equation describes a circle. If it happens that a ≠ 0, on the other hand, it describes a spiral that winds towards the origin when a < 0 and away from the origin when a > 0.

Phase portraits of linear systems, page 1

In order to summarise our results, we shall now consider six cases.

- Suppose that A has two distinct eigenvalues of the same sign. Then the phase portrait contains two lines and some curves that look like parabolas. In this case, we say that the origin is an improper node.
- Suppose that A has two distinct eigenvalues of opposite sign. Then the phase portrait contains two lines and some curves that look like hyperbolas. In this case, we say that the origin is a saddle point.
- **3** Suppose that A is diagonalisable but has a single eigenvalue $\lambda \neq 0$. Then there exists a matrix B such that $B^{-1}AB$ is diagonal and

$$B^{-1}AB = \lambda I \implies A = \lambda BB^{-1} \implies A = \lambda I.$$

This is obviously a very special case. In fact, every nonzero vector is an eigenvector of A and the phase diagram consists of lines through the origin. In this case, we say that the origin is a proper node.

Phase portraits of linear systems, page 2

- ④ Suppose that A is non-diagonalisable with a single eigenvalue λ ≠ 0. Then the phase portrait contains a single line and some curves which are asymptotically tangent to the line at the origin. Needless to say, this case is closely related to the first case involving two eigenvalues of the same sign. Thus, one still calls the origin an improper node.
- Suppose that A has two complex eigenvalues λ = ±ib, where b > 0. Then the phase portrait consists of ellipses around the origin and we say that the origin is a centre.
- 6 Suppose that A has eigenvalues λ = a ± ib, where a ≠ 0 and b > 0. Then the phase portrait consists of spirals around the origin and we say that the origin is a spiral point. The spirals get to wind towards the origin when a < 0 and away from the origin when a > 0.

These are the only cases that may arise, if the matrix A is invertible.

Phase portraits of linear systems, page 3











8 Proper node









Definition 3.2 – Critical point

Consider the autonomous system y'(t) = f(y(t)). We say that c is a critical or equilibrium point of this system, if c is a constant solution of the system. This is the case if and only if f(c) = 0.

- For instance, the critical points of the linear system y'(t) = Ay(t) are the constant vectors c that satisfy Ac = 0. When A is invertible, the only critical point is thus the origin c = 0.
- $\bullet\,$ As another example, consider the 2×2 nonlinear system

$$x'(t) = 1 - y(t),$$
 $y'(t) = 4 - x(t)^{2}.$

Then the critical points are the points (x_0, y_0) that satisfy

$$1 - y_0 = 0, \qquad 4 - x_0^2 = 0.$$

Thus, it easily follows that the only critical points are $(\pm 2, 1)$.

Definition 3.3 – Stability and asymptotic stability

Suppose that c is a critical point of the system y'(t) = f(y(t)).

1 We say that c is stable if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that every solution of the system satisfies

$$|\boldsymbol{y}(0)-\boldsymbol{c}||<\delta\quad\Longrightarrow\quad ||\boldsymbol{y}(t)-\boldsymbol{c}||<\varepsilon\quad\text{for all }t\geq0.$$

2 We say that c is asymptotically stable, if it is stable and there exists $\delta > 0$ such that every solution of the system satisfies

$$||\boldsymbol{y}(0) - \boldsymbol{c}|| < \delta \implies \lim_{t \to \infty} \boldsymbol{y}(t) = \boldsymbol{c}.$$

• Loosely speaking, stability means that every solution which is initially close to the critical point *c* must remain close to *c* at all times.

Theorem 3.4 – Stability of linear systems

Consider the system y'(t) = Ay(t), where A is a constant matrix.

- **1** The zero solution is stable if and only if the eigenvalues of A have real part $\operatorname{Re} \lambda \leq 0$ and those with $\operatorname{Re} \lambda = 0$ are simple.
- 2 The zero solution is asymptotically stable if and only if every eigenvalue of A has real part $\operatorname{Re} \lambda < 0$.
- This theorem is closely related to the fact that every solution of the system can be written as $y(t) = e^{tA}c$ for some constant vector c.
- The entries of the matrix exponential e^{tA} involve expressions of the form $t^j e^{\lambda t}$ and those approach zero as $t \to \infty$ whenever $\operatorname{Re} \lambda < 0$.
- If one of the eigenvalues has positive real part, however, then some solutions grow exponentially and the zero solution is unstable.

Stability of linear systems: Example 1

• Let $a \in \mathbb{R}$ be a fixed parameter and consider the linear system

$$\mathbf{y}'(t) = A\mathbf{y}(t), \qquad A = \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix}$$

• The characteristic polynomial of A is easily found to be

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2a\lambda + a^2 + 1.$$

• The eigenvalues of A are the roots of this polynomial, namely

$$(\lambda - a)^2 + 1 = 0 \implies \lambda - a = \pm i \implies \lambda = a \pm i.$$

 These are complex with real part a, so the zero solution is unstable when a > 0, asymptotically stable when a < 0, and also stable but not asymptotically stable in the remaining case a = 0.

Stability of linear systems: Example 2

• Let $a \in \mathbb{R}$ be a fixed parameter and consider the linear system

$$oldsymbol{y}'(t) = Aoldsymbol{y}(t), \qquad A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$$

• In this case, the characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2a\lambda + a^2 - 1.$$

 \bullet Proceeding as before, we conclude that the eigenvalues of A are

$$(\lambda - a)^2 - 1 = 0 \implies \lambda - a = \pm 1 \implies \lambda = a \pm 1.$$

• If a < -1, then both eigenvalues are negative and the zero solution is asymptotically stable. If a > -1, then a positive eigenvalue exists and the zero solution is unstable. If a = -1, finally, then $\lambda = 0, -2$ and the zero solution is stable but not asymptotically stable.

Stability of nonlinear systems: Example 1

• We study the stability of the zero solution in the case that

$$x'(t) = -2x - xy^2, \qquad y'(t) = -2y^3 + x^2y.$$

 $\bullet\,$ The distance between the point (x(t),y(t)) and the origin is

$$r(t) = \sqrt{x(t)^2 + y(t)^2} \implies r(t)^2 = x(t)^2 + y(t)^2.$$

To show that this distance is actually decreasing, we note that

$$2r(t)r'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

= 2x(-2x - xy²) + 2y(-2y³ + x²y)
= -4x² - 2x²y² - 4y⁴ + 2x²y².

 Thus, solutions which are initially close to the origin are close to the origin at all times. This implies that the zero solution is stable.

Stability of nonlinear systems: Example 2

• We show that the zero solution is asymptotically stable when

$$x'(t) = -2x + 4xy^3, \qquad y'(t) = -y - 2x^2.$$

• In this case, we define $H(t) = x(t)^2 + y(t)^4$ and we note that

$$H'(t) = 2x(t)x'(t) + 4y(t)^{3}y'(t)$$

= 2x(-2x + 4xy^{3}) + 4y^{3}(-y - 2x^{2})
= -4x^{2} + 8x^{2}y^{3} - 4y^{4} - 8x^{2}y^{3}.

• This shows that H'(t) = -4H(t), so it easily follows that

$$H(t) = Ce^{-4t} = H(0)e^{-4t} \implies \lim_{t \to \infty} H(t) = 0.$$

• Since $H(t) = x(t)^2 + y(t)^4$ goes to zero, each of x(t), y(t) must go to zero as well. Thus, the zero solution is asymptotically stable.

Definition 3.5 – Lyapunov function

Consider the $n \times n$ system $\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t))$ in the case that $\mathbf{f}(0) = 0$. We say that $V \colon \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system, if the following properties hold in an open region R around the origin.

1
$$V(\boldsymbol{x})$$
 is continuous in R ,

$${old 0} \ V({old x}) \geq 0$$
 for all ${old x} \in R$ with equality if and only if ${old x} = 0$,

3
$$\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$$
 for all $\boldsymbol{x} \in R$.

Theorem 3.6 – First Lyapunov theorem

Consider the $n \times n$ system y'(t) = f(y(t)) in the case that f(0) = 0. If a Lyapunov function exists, then the zero solution is stable.

First Lyapunov theorem: Some comments

• The third condition in the definition of a Lyapunov function V(x) is meant to ensure that V(y(t)) is decreasing in t for every solution of the system. More precisely, one may use the chain rule to find that

$$\frac{d}{dt}V(\boldsymbol{y}(t)) = \frac{d}{dt}V(y_1(t), y_2(t), \dots, y_n(t))$$
$$= \frac{\partial V}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial V}{\partial y_2}\frac{dy_2}{dt} + \dots + \frac{\partial V}{\partial y_n}\frac{dy_n}{dt}$$
$$= \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{f}(\boldsymbol{y}(t)).$$

- When it comes to applications in physics, a very natural choice for a Lyapunov function is given by the energy. In fact, an object's energy must be either conserved or else decreasing for physical reasons.
- We shall mainly look at 2×2 systems and seek Lyapunov functions that have the form $V(x, y) = ax^2 + by^2$ for some a, b > 0.

First Lyapunov theorem: Proof, page 1

• Let $\varepsilon > 0$ be given and choose $0 < r < \varepsilon$ small enough so that

$$C_r = \{ \boldsymbol{x} \in \mathbb{R}^n : ||\boldsymbol{x}|| = r \}$$

is contained within R. Since V is continuous, it attains a minimum value m>0 over the compact set C_r and one has

$$m \le V(\boldsymbol{x})$$
 whenever $||\boldsymbol{x}|| = r.$ (L1)

• Since V is continuous at x = 0, there also exists $\delta > 0$ such that

$$||\boldsymbol{x}|| < \delta \implies V(\boldsymbol{x}) < m.$$
(L2)

Note that the last two equations imply that δ ≤ r. To prove stability, we show that every solution which satisfies ||y(0)|| < δ is actually a global solution which satisfies ||y(t)|| < r < ε for all t ≥ 0.

First Lyapunov theorem: Proof, page 2

• Now, suppose that the inequality $||\boldsymbol{y}(t)|| < r$ is violated at some point and let T be the first such point. When it comes to times $0 \le t < T$, the graph of $\boldsymbol{y}(t)$ lies within C_r , so it lies within R and one has

$$\frac{d}{dt}V(\boldsymbol{y}(t)) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{f}(\boldsymbol{y}(t)) \leq 0.$$

• In particular, $V(\boldsymbol{y}(t))$ is decreasing on [0,T) and (L2) implies that

$$||\boldsymbol{y}(0)|| < \delta \implies V(\boldsymbol{y}(t)) \le V(\boldsymbol{y}(0)) < m$$

for all $0 \le t < T$. Taking the limit as $t \to T$, we conclude that

$$V(\boldsymbol{y}(T)) \le V(\boldsymbol{y}(0)) < m.$$

• However, the last equation contradicts (L1) because $||\boldsymbol{y}(T)|| = r$ by definition. Thus, the inequality $||\boldsymbol{y}(t)|| < r$ must hold at all times.

First Lyapunov theorem: Example 1

• We show that the zero solution is stable in the case that

$$x'(t) = y^2 - x^3, \qquad y'(t) = -y - 2xy.$$

• Consider the function $V(x, y) = ax^2 + by^2$ for some a, b > 0. This is certainly continuous and non-negative, while V(x, y) = 0 if and only if x = y = 0. To show that V is a Lyapunov function, we compute

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)$$

= $2ax(y^2 - x^3) + 2by(-y - 2xy)$
= $(2a - 4b)xy^2 - 2ax^4 - 2by^2$.

• We need to ensure that $\nabla V \cdot f \leq 0$ and this is obviously true, if we let a = 2b > 0. In other words, $V(x, y) = 2bx^2 + by^2$ is a Lyapunov function for any constant b > 0, so the zero solution is stable.

First Lyapunov theorem: Example 2

• We show that the zero solution is stable in the case that

$$x'(t) = y^2 - 2x, \qquad y'(t) = x^2 - y.$$

• To see that $V(x,y) = x^2 + y^2$ is a Lyapunov function, we note that

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)$$
$$= 2x(y^2 - 2x) + 2y(x^2 - y)$$
$$= (2x - 2)y^2 + (2y - 4)x^2.$$

- This expression is not negative at all points, but it is negative near the origin since both 2x 2 and 2y 4 are negative near the origin.
- Consider the region R = (-1,1) × (-2,2), for instance. Since this is open and one has ∇V · f ≤ 0 within R, the zero solution is stable.

Pendulum equation, page 1

- Consider a pendulum consisting of a mass m which is attached to a massless string of length L. To describe its motion, we look at the angle θ(t) formed between the pendulum and the vertical axis.
- Since the pendulum moves along a circle of radius L, its position at time t is the length of the arc corresponding to the angle θ, namely

Position = Arc length =
$$L \cdot \theta(t)$$
.

- The only force acting on the pendulum is the gravitational force and its component in the direction of motion has length $-mg\sin\theta(t)$.
- According to Newton's second law of motion, we must thus have

$$m(L\theta)'' = -mg\sin\theta \implies \theta'' = -\frac{g}{L}\sin\theta.$$

• This nonlinear equation is also known as the pendulum equation.

Pendulum equation, page 2

• Let us now express the pendulum equation as a 2×2 system. If we introduce the variables $x = \theta(t)$ and $y = \theta'(t)$, we can then write

$$x' = \theta' = y,$$
 $y' = \theta'' = -\frac{g}{L}\sin x.$

• The zero solution x = y = 0 corresponds to the equilibrium vertical position and it ought to be stable for physical reasons. In order to prove its stability, we shall now consider the function

$$V(x,y) = \frac{1}{2}mL^{2}y^{2} + mgL(1 - \cos x).$$

- This function actually represents the pendulum's energy. It is the sum of the kinetic energy $\frac{1}{2}mv^2$ and the potential energy mgh, where v is the velocity and h is the height above the equilibrium position.
- We need to show that V(x, y) is a Lyapunov function for the system.

Pendulum equation, page 3

• Consider the region $R = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$. This is obviously open, while

$$V(x,y) = \frac{1}{2}mL^{2}y^{2} + mgL(1 - \cos x)$$

is continuous and non-negative. Since $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we also have

$$V(x,y)=0 \iff y=0 \text{ and } \cos x=1 \iff x=y=0.$$

This proves the first two properties of a Lyapunov function, while

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)$$
$$= mgL \sin x \cdot y + mL^2 y \cdot (-g \sin x)/L = 0.$$

• Thus, V(x, y) is a Lyapunov function and the zero solution is stable.

Definition 3.7 – Strict Lyapunov function

Consider the $n \times n$ system $\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t))$ in the case that $\mathbf{f}(0) = 0$. We say that $V \colon \mathbb{R}^n \to \mathbb{R}$ is a strict Lyapunov function for the system, if the following properties hold in an open region R around the origin.

$${f 0}~V({m x})$$
 is continuously differentiable in R ,

$$oldsymbol{ ilde O}$$
 $V(oldsymbol{x})\geq 0$ for all $oldsymbol{x}\in R$ with equality if and only if $oldsymbol{x}=0$,

3 $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$ in R with equality if and only if $\boldsymbol{x} = 0$.

Theorem 3.8 – Second Lyapunov theorem

Consider the $n \times n$ system y'(t) = f(y(t)) when f(0) = 0. If there is a strict Lyapunov function, the zero solution is asymptotically stable.

Second Lyapunov theorem: Proof, page 1

• Choose $\varepsilon > 0$ so that the sphere $||\mathbf{x}|| = \varepsilon$ is contained within R. Since the zero solution is stable, there exists $\delta > 0$ such that

$$||\boldsymbol{y}(0)|| < \delta \implies ||\boldsymbol{y}(t)|| < \varepsilon \text{ for all } t \ge 0.$$

• We note that $V(\boldsymbol{y}(t))$ is non-negative and also decreasing since

$$\frac{d}{dt}V(\boldsymbol{y}(t)) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{f}(\boldsymbol{y}(t)) \leq 0.$$

• Thus, $V(\boldsymbol{y}(t))$ attains a limit $L \geq 0$ as $t \to \infty$. If L = 0, then

$$\lim_{t \to \infty} V(\boldsymbol{y}(t)) = 0 \quad \Longrightarrow \quad \lim_{t \to \infty} \boldsymbol{y}(t) = 0$$

and asymptotic stability follows. To finish the proof, it remains to show that the remaining case L > 0 leads to a contradiction.

Second Lyapunov theorem: Proof, page 2

• Suppose now that L > 0. Then there exists $\gamma > 0$ such that

$$||\boldsymbol{x}|| < \gamma \implies V(\boldsymbol{x}) < L.$$

• Since we also have $V(\boldsymbol{y}(t)) \geq L$ at all times, the solution must lie in the annulus $\gamma \leq ||\boldsymbol{y}(t)|| \leq \varepsilon$ at all times. Letting m be the maximum value attained by $\nabla V \cdot \boldsymbol{f}$ in this annulus, we find that m < 0 and

$$\frac{d}{dt}V(\boldsymbol{y}(t)) = \nabla V(\boldsymbol{y}(t)) \cdot \boldsymbol{f}(\boldsymbol{y}(t)) \le m.$$

• Integrating over the interval [0, t], we may thus conclude that

$$V(\boldsymbol{y}(t)) \leq V(\boldsymbol{y}(0)) + mt \quad \Longrightarrow \quad \lim_{t \to \infty} V(\boldsymbol{y}(t)) = -\infty.$$

• This obviously contradicts the fact that $V(\boldsymbol{y}(t)) \ge 0$ at all times.

Second Lyapunov theorem: Example 1

• We show that the zero solution is asymptotically stable when

$$x'(t) = y^3 - x, \qquad y'(t) = -y - 3xy^2.$$

 \bullet Consider the function $V(x,y)=ax^2+by^2$ and note that

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)$$

= $2ax(y^3 - x) + 2by(-y - 3xy^2)$
= $(2a - 6b)xy^3 - 2ax^2 - 2by^2.$

• If we now let 0 < a = 3b, then the last equation ensures that

$$\nabla V \cdot \boldsymbol{f} = -6bx^2 - 2by^2 \le 0$$

with equality only at the origin. Thus, $V(x, y) = 3bx^2 + by^2$ is a strict Lyapunov function and the zero solution is asymptotically stable.

Second Lyapunov theorem: Example 2

• Let a > 0 be a fixed parameter and consider the system

$$x'(t) = ay^2 - x, \qquad y'(t) = x^2 - 2y.$$

 \bullet We define the function $V(x,y)=x^2+y^2$ and we note that

$$\nabla V \cdot \boldsymbol{f} = \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)$$
$$= 2x(ay^2 - x) + 2y(x^2 - 2y)$$
$$= (2y - 2)x^2 + (2ax - 4)y^2.$$

• This expression is negative within $R = \left(-\frac{1}{a}, \frac{1}{a}\right) imes \left(-\frac{1}{2}, \frac{1}{2}\right)$ because

$$\nabla V \cdot \boldsymbol{f} \le (1-2)x^2 + (2-4)y^2 \le 0$$

within R. In fact, $\nabla V \cdot f = 0$ only at the origin, so V(x, y) is a strict Lyapunov function and the zero solution is asymptotically stable.

Theorem 3.9 – Linear approximation

Suppose that c is a critical point of the system y'(t) = f(y(t)) and that f is continuously differentiable. Then z(t) = y(t) - c satisfies

$$\boldsymbol{z}'(t) = A \cdot \boldsymbol{z}(t) + R(t) \cdot \boldsymbol{z}(t),$$

where A is the Jacobian matrix with entries $a_{ij} = \frac{\partial f_i}{\partial y_j}(c)$ and R(t) is a matrix whose entries go to zero as $||\boldsymbol{z}||$ goes to zero.

- In other words, the difference z(t) = y(t) c satisfies a system that behaves like the linear system z' = Az near the critical point c.
- This suggests that the stability of the critical point *c* is closely related to the eigenvalues of the Jacobian matrix *A*. If all the eigenvalues are negative, for instance, then *z*(*t*) behaves like a linear combination of decaying exponentials, so one expects *c* to be asymptotically stable.

Theorem 3.10 – Linearisation method

Suppose c is a critical point of the system y'(t) = f(y(t)) and f is continuously differentiable. Let A be the Jacobian matrix as before.

- If every eigenvalue of A has negative real part, then the critical point c is both stable and asymptotically stable.
- ${\it 2}$ If some eigenvalue of A has positive real part, then c is unstable.

• We shall mainly be concerned with 2×2 systems of the form

$$x'(t) = f(x, y), \qquad y'(t) = g(x, y).$$

• In that case, the Jacobian matrix at a critical point (x_0, y_0) is

$$A = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}.$$

Linearisation method: Example 1, page 1

 $\bullet\,$ Consider the critical points of the 2×2 nonlinear system

$$x'(t) = x^2 - y,$$
 $y'(t) = x - y.$

• To find them explicitly, one needs to solve the equations

$$x^2 - y = x - y = 0 \implies x = y, \qquad x^2 - x = 0.$$

It easily follows that the critical points are P(0,0) and Q(1,1).

• To study their stability properties, we look at the Jacobian matrix

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2x & -1 \\ 1 & -1 \end{bmatrix}$$

and we compute its eigenvalues. Since the eigenvalues depend on x, one needs to examine each of the critical points separately.

Linearisation method: Example 1, page 2

• When it comes to the critical point P(0,0), the Jacobian matrix is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

and its eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 + \lambda + 1 = 0 \implies \lambda = \frac{-1 \pm i\sqrt{3}}{2}.$$

Thus, their real part is negative and P is asymptotically stable. • When it comes to the critical point Q(1,1), the Jacobian matrix is

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

and its eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 1 = 0 \implies \lambda = \frac{1 \pm \sqrt{5}}{2}.$$

In particular, one of the eigenvalues is positive and Q is unstable.

Linearisation method: Example 2, page 1

• We study the critical points of the 2×2 nonlinear system

$$x'(t) = x(y-1),$$
 $y'(t) = x - y - 1.$

• To find them explicitly, one needs to solve the equations

$$x(y-1) = 0, \qquad x = y+1.$$

When x = 0, the second equation gives y = -1. When y = 1, the second equation gives x = 2. This means that the system has two critical points, namely R(0, -1) and S(2, 1).

• To study their stability properties, we look at the Jacobian matrix

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y - 1 & x \\ 1 & -1 \end{bmatrix}$$

and we compute the eigenvalues of A for each of the critical points.

Linearisation method: Example 2, page 2

• When it comes to the critical point R(0,-1), the Jacobian matrix is

$$A = \begin{bmatrix} -2 & 0\\ 1 & -1 \end{bmatrix}.$$

This is lower triangular, so its eigenvalues are the diagonal entries. As those are both negative, we conclude that R is asymptotically stable.

• When it comes to the critical point S(2,1), the Jacobian matrix is

$$A = \begin{bmatrix} 0 & 2\\ 1 & -1 \end{bmatrix}$$

and its eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 + \lambda - 2 = 0 \implies \lambda = \frac{-1 \pm 3}{2} = -2, 1.$$

In particular, one of the eigenvalues is positive and S is unstable.