A linear homogeneous system is a system that has the form

$$y'(t) = A(t)y(t),$$  \hspace{1cm} \text{(LHS)}$$

where $y(t)$ is the vector of unknowns and $A(t)$ is a square matrix.

The set of solutions of a linear homogeneous system is closed under addition and scalar multiplication. In other words, the sum of any two solutions is a solution and scalar multiples of solutions are solutions.

- The superposition principle asserts that the solutions of (LHS) form a vector space. If one can find some solutions that form a basis for this vector space, then every solution will be a linear combination of them.
Linear independence of functions

**Definition 2.3 – Linear independence of functions**

The functions \( y_1(t), y_2(t), \ldots, y_n(t) \) are called linearly dependent, if there exist constants \( c_1, c_2, \ldots, c_n \) which are not all zero such that

\[
c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) = 0 \quad \text{for all } t.
\]

Otherwise, we say that the functions are linearly independent.

- Linear independence of vector-valued functions is a bit more subtle than linear independence of constant vectors. This is because the coefficients \( c_k \) are not allowed to depend on \( t \). For instance,

\[
y_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}
\]

are linearly independent functions, even though one has \( y_2 = ty_1 \).
Linear independence of functions: Example

- We check that \( y_1(t), y_2(t), y_3(t) \) are linearly independent when

\[
y_1(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} t \\ t \end{bmatrix}, \quad y_3(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

- Suppose that \( c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0 \), in which case

\[
c_1 e^t + c_2 t + c_3 = c_1 t + c_2 t + c_3 = 0 \quad \text{for all } t.
\]

One may analyse this relation by considering special values of \( t \) or by differentiating, for instance. Differentiating twice, we get \( c_1 e^t = 0 \) for all \( t \), hence also \( c_1 = 0 \). The given relation may thus be reduced to

\[
c_2 t + c_3 = 0 \quad \text{for all } t.
\]

- Letting \( t = 0 \) and \( t = 1 \), we now get \( c_3 = 0 = c_2 + c_3 \). This implies that \( c_k = 0 \) for all \( k \), so the given functions are linearly independent.
Linear independence of solutions

**Theorem 2.4 – Linear independence of solutions**

Suppose that \( y_1(t), y_2(t), \ldots, y_n(t) \) are solutions of the \( n \times n \) system

\[
y'(t) = A(t)y(t). \tag{LHS}
\]

Then \( y_1(t), y_2(t), \ldots, y_n(t) \) are linearly independent functions if and only if \( y_1(0), y_2(0), \ldots, y_n(0) \) are linearly independent vectors.

- The solutions of an \( n \times n \) linear homogeneous system form a vector space of dimension \( n \). In fact, let \( v_1, v_2, \ldots, v_n \) be any basis of \( \mathbb{R}^n \) and let \( y_k(t) \) be the unique solution of the initial value problem

  \[
y_k'(t) = A(t)y_k(t), \quad y_k(0) = v_k.
\]

- Then \( y_1(t), y_2(t), \ldots, y_n(t) \) are easily seen to form a basis for the space of solutions. However, such a basis is not usually explicit.
We obtain a basis of solutions for the linear homogeneous system

\[ y'(t) = A(t)y(t), \quad A(t) = \begin{bmatrix} 1 & 0 \\ e^t & 2 \end{bmatrix}. \]

In this case, \( A(t) \) is lower triangular, so it is easier to look at the corresponding equations one by one. Let us start by writing

\[ x'(t) = x(t), \quad y'(t) = e^t x(t) + 2y(t). \]

When it comes to the leftmost equation, one clearly has

\[ x'(t) = x(t) \quad \implies \quad x(t) = c_1 e^t. \]

We now insert this fact in the rightmost equation to find that

\[ y'(t) - 2y(t) = e^t x(t) = c_1 e^{2t}. \]

This is a first-order linear equation with integrating factor \( \mu = e^{-2t} \).
Multiplying by the integrating factor, we conclude that

\[(e^{-2t}y)' = c_1 \implies e^{-2t}y(t) = c_1 t + c_2 \]

\[\implies y(t) = (c_1 t + c_2)e^{2t}.\]

This shows that every solution of the system has the form

\[y(t) = \begin{bmatrix} c_1 e^t \\ (c_1 t + c_2)e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}.\]

In other words, every solution is a linear combination of

\[y_1(t) = \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}\]

and these functions form a basis for the space of solutions.
When \( A \) is constant, the linear system \( y'(t) = Ay(t) \) can always be solved explicitly by relating \( A \) to its Jordan form, say \( J = B^{-1}AB \).

More precisely, the change of variables \( z(t) = B^{-1}y(t) \) gives

\[
z'(t) = B^{-1}y'(t) = B^{-1}Ay(t) = B^{-1}ABz(t) = Jz(t).
\]

This is a linear system that involves a lower triangular matrix, while each of the corresponding equations has the form

\[
z'_k = \lambda_k z_k \quad \text{or} \quad z'_k = z_{k-1} + \lambda_k z_k.
\]

In particular, each of these equations is first-order linear and one may determine the variables \( z_k \) inductively using integrating factors.

The corresponding formula for the solution \( y(t) = Bz(t) \) turns out to be simple when \( A \) is diagonalisable but a bit technical, otherwise. We shall thus use another approach to deal with the general case later.
Consider the $n \times n$ linear system $y'(t) = Ay(t)$ in the case that $A$ is constant and diagonalisable. Let $v_1, v_2, \ldots, v_n$ be linearly independent eigenvectors and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues. Then every solution of the system has the form

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \ldots + c_n e^{\lambda_n t} v_n.$$  

- The coefficients $c_k$ may be taken to be real, if the eigenvalues of $A$ are all real. Otherwise, the coefficients $c_k$ will generally be complex.
- To prove this theorem, we note that each $y_k(t) = e^{\lambda_k t} v_k$ satisfies

$$y_k'(t) = \lambda_k e^{\lambda_k t} v_k = e^{\lambda_k t} Av_k = Ay_k(t).$$

This gives $n$ solutions which are linearly independent when $t = 0$, so every other solution must be a linear combination of them.
Eigenvector method: Example 1

- We use the eigenvector method to solve the linear system
  
  \[ y'(t) = Ay(t), \quad A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}. \]

- The eigenvalues of \( A \) are the roots of the characteristic polynomial
  
  \[ f(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1), \]
  
  namely \( \lambda_1 = 7 \) and \( \lambda_2 = 1 \). These correspond to the eigenvectors

  \[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

- According to the previous theorem, the solution of the system is

  \[ y(t) = c_1 e^{7t} \mathbf{v}_1 + c_2 e^t \mathbf{v}_2 = \begin{bmatrix} c_1 e^{7t} + c_2 e^t \\ 2c_1 e^{7t} - c_2 e^t \end{bmatrix}. \]
Eigenvector method: Example 2

- We use the eigenvector method to solve the linear system

\[ y'(t) = Ay(t), \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}. \]

- Since \( A \) is lower triangular, its eigenvalues \( \lambda = 1, 3, 4 \) are merely the diagonal entries of \( A \). These are distinct, so \( A \) is diagonalisable and one may easily check that the corresponding eigenvectors are

\[ \mathbf{v}_1 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

- In view of the previous theorem, the solution of the system is thus

\[ y(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{3t} \mathbf{v}_2 + c_3 e^{4t} \mathbf{v}_3 = \begin{bmatrix} -3c_1 e^t \\ 3c_1 e^t - c_2 e^{3t} \\ c_1 e^t + c_2 e^{3t} + c_3 e^{4t} \end{bmatrix}. \]
Given a square matrix $A$, we define its exponential $e^A$ as the series

$$e^A = I + A + \frac{1}{2!} A^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$ 

It can be shown that this series converges for every square matrix $A$.

- To compute the powers of a square matrix, one relates them to the powers of its Jordan form $J = B^{-1} A B$ using the computation

$$A^k = (BJB^{-1})^k = BJ^k B^{-1}.$$ 

- A similar approach can be used for the exponential of $A$ since

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} BJ^k B^{-1} = Be^J B^{-1}.$$
Matrix exponential: Properties

**Theorem 2.7 – Properties of the matrix exponential**

Suppose $A, B$ are $n \times n$ matrices and let $\Phi(t) = e^{tA}$ for all $t \in \mathbb{R}$.

1. The exponential property $e^{A+B} = e^A e^B$ holds when $AB = BA$, but this property is generally false for arbitrary matrices.
2. The exponential function $\Phi(t) = e^{tA}$ is such that $\Phi'(t) = A\Phi(t)$. In particular, it is a matrix solution of the system $y'(t) = Ay(t)$.
3. The columns of $\Phi(t) = e^{tA}$ are vector solutions of $y'(t) = Ay(t)$ and they also form a basis for the space of all solutions.

- The second property may be stated simply as $(e^{tA})' = Ae^{tA}$. This resembles the chain rule for the standard exponential function.
- The product rule $(AB)' = A'B + AB'$ also holds for matrix-valued functions, but the chain rule $(A^2)' = 2AA'$ is generally false.
Matrix exponential: Jordan forms

**Theorem 2.8 – Matrix exponential of a Jordan form**

Suppose that $J$ is a $k \times k$ Jordan block with eigenvalue $\lambda$. Then the exponential $e^{tJ}$ is a lower triangular matrix and the entries that lie $i$ steps below the diagonal are equal to $\frac{t^j}{j!}e^{\lambda t}$ for each $0 \leq j < k$.

- For instance, the exponential of a $3 \times 3$ Jordan block is given by

\[
J = \begin{pmatrix}
\lambda & 1 & \\
1 & \lambda & \\
1 & 1 & \lambda
\end{pmatrix} \implies e^{tJ} = \begin{pmatrix}
e^{\lambda t} & te^{\lambda t} & e^{\lambda t} \\
t e^{\lambda t} & e^{\lambda t} & \\
\frac{t^2}{2}e^{\lambda t} & te^{\lambda t} & e^{\lambda t}
\end{pmatrix}.
\]

- The exponential of a Jordan form is obtained by exponentiating each Jordan block separately. As a typical example, one has

\[
J = \begin{pmatrix}
2 & & \\
3 & & \\
1 & 3 & 
\end{pmatrix} \implies e^{tJ} = \begin{pmatrix}
e^{2t} & & \\
e^{3t} & & \\
te^{3t} & e^{3t} & 
\end{pmatrix}.
\]
We compute the matrix exponential of the diagonalisable matrix

\[ A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}. \]

The characteristic polynomial of this matrix is given by

\[ f(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5), \]

so the eigenvalues are real and distinct, namely \( \lambda_1 = 2 \) and \( \lambda_2 = 5 \).

The corresponding eigenvectors are easily found to be

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Once we now merge the eigenvectors to form a matrix \( B \), we get

\[ B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \implies \quad J = B^{-1}AB = \begin{bmatrix} 2 \\ 5 \end{bmatrix}. \]
Since the Jordan form \( J \) is diagonal, the same is true for \( e^{tJ} \) and

\[
J = B^{-1}AB = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \implies e^{tJ} = \begin{bmatrix} e^{2t} \\ e^{5t} \end{bmatrix}.
\]

As for the exponential of the original matrix \( A \), this is given by

\[
J = B^{-1}AB \implies A = BJB^{-1} \implies e^{tA} = Be^{tJ}B^{-1}.
\]

In view of our computations above, we must thus have

\[
e^{tA} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{5t} \end{bmatrix} \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}
= \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{5t} & -e^{2t} + e^{5t} \\ -2e^{2t} + 2e^{5t} & 2e^{2t} + e^{5t} \end{bmatrix}.
\]

The exact same approach applies for any diagonalisable matrix \( A \).
We compute the matrix exponential of the non-diagonalisable matrix

\[ A = \begin{bmatrix} 9 & -4 \\ 9 & -3 \end{bmatrix}. \]

In this case, the characteristic polynomial of \( A \) is given by

\[ f(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]

so the only eigenvalue is \( \lambda = 3 \). The only eigenvector turns out to be

\[ \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

This implies that \( A \) is not diagonalisable and that the Jordan form is

\[ J = B^{-1}AB = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \Rightarrow \quad e^{tJ} = \begin{bmatrix} e^{3t} & te^{3t} \\ te^{3t} & e^{3t} \end{bmatrix}. \]

Let us now find a matrix \( B \) such that \( J = B^{-1}AB \) is in Jordan form.
Pick any nonzero vector $v_1$ which is not an eigenvector and let

$$v_2 = (A - \lambda I)v_1, \quad B = \begin{bmatrix} v_1 & v_2 \end{bmatrix}.$$

There are obviously infinitely many choices and one possibility is

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = (A - 3I)v_1 = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 0 & 9 \end{bmatrix}.$$

In view of our computations above, we must thus have

$$e^{tA} = B e^{tJ} B^{-1} = \begin{bmatrix} 1 & 6 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} e^{3t} & te^{3t} \\ te^{3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -2/3 \\ 0 & 1/9 \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} 1 + 6t & -4t \\ 9t & 1 - 6t \end{bmatrix}.$$

This approach applies for any non-diagonalisable $2 \times 2$ matrix $A$. 
Finally, we consider a real matrix \( A \) with complex eigenvalues, say
\[
A = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}.
\]

The characteristic polynomial of this matrix is easily found to be
\[
f(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1.
\]

The eigenvalues \( \lambda = 1 \pm i \) are complex conjugates of one another and the same is true for the corresponding eigenvectors which are given by
\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.
\]

This implies that \( A \) is diagonalisable and that we also have
\[
B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 1 + i & 1 \\ 1 - i & 1 \end{bmatrix}.
\]

In particular, one may proceed as before to compute \( e^{tJ} \) and then \( e^{tA} \).
Since the Jordan form $J$ is diagonal, the same is true for $e^{tJ}$ and 

$$J = B^{-1}AB = \begin{bmatrix} 1 + i & 1 \\ 1 - i & 1 \end{bmatrix} \quad \implies \quad e^{tJ} = \begin{bmatrix} e^{t}e^{it} & \cdot \\ \cdot & e^{t}e^{-it} \end{bmatrix}.$$ 

In view of our computations above, we must thus have

$$e^{tA} = B e^{tJ} B^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{t}e^{it} & \cdot \\ \cdot & e^{t}e^{-it} \end{bmatrix} \begin{bmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{bmatrix}$$

$$= \frac{e^{t}}{2} \begin{bmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ i(e^{-it} - e^{it}) & e^{it} + e^{-it} \end{bmatrix}.$$ 

On the other hand, one has $e^{\pm it} = \cos t \pm i \sin t$, so this implies that

$$e^{tA} = \frac{e^{t}}{2} \begin{bmatrix} 2 \cos t & -2 \sin t \\ 2 \sin t & 2 \cos t \end{bmatrix} = \begin{bmatrix} e^{t} \cos t & -e^{t} \sin t \\ e^{t} \sin t & e^{t} \cos t \end{bmatrix}.$$ 

Needless to say, $e^{tA}$ will always turn out to be real when $A$ is real.
### Definition 2.9 – Fundamental matrix

We say that $\Phi(t)$ is a fundamental matrix for a linear homogeneous system, if the columns of $\Phi(t)$ form a basis for the space of solutions.

- The most common example is the matrix exponential $\Phi(t) = e^{tA}$. It is a fundamental matrix for the system $y'(t) = Ay(t)$, if $A$ is constant.

### Theorem 2.10 – Properties of fundamental matrices

Let $\Phi(t)$ be a fundamental matrix for the system $y'(t) = A(t)y(t)$.

1. Every solution is a linear combination of the columns of $\Phi(t)$.
2. Every solution has the form $y(t) = \Phi(t)c$ for some vector $c$.
3. The fundamental matrix itself is a matrix solution of the system. In other words, one has the matrix identity $\Phi'(t) = A(t)\Phi(t)$. 

It is only in a few special cases that one may explicitly determine a fundamental matrix for the linear system $y'(t) = A(t)y(t)$.

When $A(t)$ is either upper or lower triangular, the system can be solved explicitly by solving the corresponding equations one by one. Let $y_k(t)$ be the unique solution of the initial value problem

$$y'_k(t) = A(t)y_k(t), \quad y_k(0) = e_k.$$ 

Then $y_1(t), y_2(t), \ldots, y_n(t)$ form a basis for the space of solutions.

When $A(t)$ is a matrix that commutes with its antiderivative $B(t)$, a fundamental matrix for the system is given by

$$\Phi(t) = e^{B(t)}, \quad B(t) = \int_0^t A(s) \, ds.$$ 

This is the case, in particular, when $A(t) = A$ is a constant matrix.
Let us now turn our attention to the linear inhomogeneous system

\[
y'(t) = A(t)y(t) + b(t). \tag{LIS}
\]

When it comes to the special case \( b(t) = 0 \), there is a fundamental matrix \( \Phi(t) \) which satisfies the identity \( \Phi'(t) = A(t)\Phi(t) \) and every solution has the form \( y(t) = \Phi(t)c \) for some constant vector \( c \).

To deal with the general case, we look for solutions that have the form \( y(t) = \Phi(t)c(t) \), where \( c(t) \) is not necessarily constant. Since

\[
y'(t) = \Phi'(t)c(t) + \Phi(t)c'(t)
\]

\[
= A(t)\Phi(t)c(t) + \Phi(t)c'(t)
\]

\[
= A(t)y(t) + \Phi(t)c'(t),
\]

we do obtain a solution of (LIS), provided that \( \Phi(t)c'(t) = b(t) \).

Thus, one may use \( \Phi(t) \) to solve the inhomogeneous system as well.
Theorem 2.11 – Variation of parameters

Consider the linear inhomogeneous system

\[ y'(t) = A(t)y(t) + b(t). \]  \hfill (LIS)

If \( A(t) \) and \( b(t) \) are continuous, then every solution has the form

\[ y(t) = \Phi(t)c + \Phi(t) \int \Phi(t)^{-1}b(t) \, dt, \]

where \( c \) is a constant vector and \( \Phi(t) \) is a fundamental matrix for the associated linear homogeneous system \( y'(t) = A(t)y(t) \).

The integral term in the equation above is itself a particular solution of the system. According to the theorem, every solution is thus the sum of the homogeneous solution \( \Phi(t)c \) and a particular solution.
We use variation of parameters to solve the inhomogeneous system

$$y'(t) = Ay(t) + b(t), \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad b(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

Since $A$ is constant, a fundamental matrix is $\Phi(t) = e^{tA}$ and thus

$$y(t) = e^{tA}c + e^{tA} \int e^{-tA}b(t) \, dt$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \, dt$$

$$= \begin{bmatrix} c_1e^t \\ c_1te^t + c_2e^t \end{bmatrix} + \begin{bmatrix} c_1te^t + c_2e^t \\ c_1te^t + c_2e^t + te^t \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} \, dt$$

$$= \begin{bmatrix} c_1e^t - 1 \\ c_1te^t + c_2e^t - t \end{bmatrix}.$$
Suppose that we need to solve a scalar linear equation such as

\[ y'''(t) - 5y''(t) + 7y'(t) - 3y(t) = 0.\]

This is a 3rd-order equation, so one may express it as a $3 \times 3$ system.

More precisely, let $y$ be the vector with entries $y, y', y''$ and note that

\[
\begin{bmatrix}
y \\
y' \\
y''
\end{bmatrix}
\implies
\begin{bmatrix}
y' \\
y'' \\
y'''
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
3 & -7 & 5
\end{bmatrix}
\begin{bmatrix}
y \\
y' \\
y''
\end{bmatrix}
= Ay.
\]

Since the scalar equation is linear, the same is true for the system, so one may determine $y$ using methods we have already developed.

This kind of approach is certainly valid, but it is not very efficient, as we are only interested in the first entry of $y$. It is thus worth having some related results that deal with scalar equations directly.
Theorem 2.12 – Linear homogeneous equations

Consider the scalar linear homogeneous equation

\[ a_n y^{(n)}(t) + \ldots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0. \]  

(LHE)

If the coefficients \(a_k\) are all constant, then one may obtain a basis of solutions by solving the corresponding characteristic equation

\[ a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \]

and by associating each root \(\lambda\) with solutions of (LHE) as follows.

1. If a real root \(\lambda\) has multiplicity \(k\), it gets associated with the \(k\) functions \(\{t^j e^{\lambda t}\}_{j=0}^{k-1}\), namely with \(e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1} e^{\lambda t}\).

2. If a pair of complex roots \(\lambda = a \pm bi\) has multiplicity \(k\), it gets associated with the \(2k\) functions \(\{t^j e^{at} \sin(bt), t^j e^{at} \cos(bt)\}_{j=0}^{k-1}\).
We use the previous theorem to solve the homogeneous equation
\[ y'''(t) - 5y''(t) + 7y'(t) - 3y(t) = 0. \]

In this case, the associated characteristic equation is given by
\[ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0. \]

Noting that \( \lambda = 1 \) is a root, one easily finds that
\[ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = (\lambda - 1)(\lambda^2 - 4\lambda + 3) = (\lambda - 1)^2(\lambda - 3). \]

This means that \( \lambda = 1 \) is a double root, while \( \lambda = 3 \) is a simple root. Thus, a basis of solutions is formed by the functions \( e^t, te^t, e^{3t} \) and every solution of the given equation has the form
\[ y(t) = c_1 e^t + c_2 te^t + c_3 e^{3t}. \]
Let us now solve an initial value problem such as

$$y''(t) - y(t) = 0, \quad y(0) = 1, \quad y'(0) = 3.$$  

In this case, the associated characteristic equation gives

$$\lambda^2 - 1 = 0 \implies (\lambda + 1)(\lambda - 1) = 0 \implies \lambda = -1, 1.$$  

Since the roots are both simple, every solution has the form

$$y(t) = c_1 e^t + c_2 e^{-t}.$$  

Next, we turn to the initial conditions and we note that

$$y(t) = c_1 e^t + c_2 e^{-t} \implies 1 = y(0) = c_1 + c_2,$$
$$y'(t) = c_1 e^t - c_2 e^{-t} \implies 3 = y'(0) = c_1 - c_2.$$  

Solving this system of equations, we find that $c_1 = 2$ and $c_2 = -1$. Thus, the unique solution is given by $y(t) = 2e^t - e^{-t}$. 
The equation that describes a simple harmonic oscillator is

\[ m\ddot{y}(t) = -ky(t). \]

Here, the constants \( k, m \) are both positive, so one may also write

\[ \ddot{y}(t) + \omega^2 y(t) = 0, \quad \omega = \sqrt{k/m}. \]

Solving the associated characteristic equation, we now get

\[ \lambda^2 + \omega^2 = 0 \quad \implies \quad \lambda^2 = -\omega^2 \quad \implies \quad \lambda = \pm i\omega. \]

This is a pair of complex roots, so every solution has the form

\[ y(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t). \]

In particular, every solution of the given equation is periodic.
As our last example on homogeneous equations, we now solve
\[ y'''(t) + 7y''(t) + 19y'(t) + 13y(t) = 0. \]

In this case, the associated characteristic equation is given by
\[ \lambda^3 + 7\lambda^2 + 19\lambda + 13 = 0. \]

Noting that \( \lambda = -1 \) is a root, one may factor the cubic as
\[ \lambda^3 + 7\lambda^2 + 19\lambda + 13 = (\lambda + 1)(\lambda^2 + 6\lambda + 13). \]

The roots of the quadratic factor are easily found to be
\[ \lambda = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 13}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i. \]

We may thus conclude that every solution has the form
\[ y(t) = c_1 e^{-t} + c_2 e^{-3t} \sin(2t) + c_3 e^{-3t} \cos(2t). \]
Suppose that we need to solve an inhomogeneous equation such as
\[ y''(t) - 3y'(t) + 2y(t) = 2t + 5. \]

The solution of such an equation can be expressed as the sum of the homogeneous solution \( y_h \) and a particular solution \( y_p \). More precisely, the difference \( z = y - y_p \) between any two solutions satisfies
\[ z''(t) - 3z'(t) + 2z(t) = 0, \]
so it is a solution of the corresponding homogeneous equation.

This proves the useful identity \( y = y_h + y_p \). We already know how to find the homogeneous solution \( y_h \), so we need only worry about \( y_p \).

There are two methods for finding a particular solution: the method of undetermined coefficients and variation of parameters. The former is generally simpler, but it only applies in a few special cases.
Undetermined coefficients: Main result

**Theorem 2.13 – Undetermined coefficients**

Consider the scalar linear inhomogeneous equation

\[ a_n y^{(n)}(t) + \ldots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t). \]  

Suppose that the coefficients \( a_k \) are all constant and that the right hand side \( f(t) \) is a linear combination of terms that have the form

\[ t^j e^{\lambda t}, \quad t^j e^{at} \sin(bt), \quad t^j e^{at} \cos(bt). \]

Then the solution \( y(t) \) satisfies a higher-order homogeneous equation, so it can itself be expressed as a linear combination of such terms.

One typically uses this theorem to write down an explicit formula for a particular solution \( y_p \). It is easy to predict the terms that appear in the formula, but their exact coefficients need to be determined.
The general rules for finding a particular solution $y_p$ are the following.

1. If $f(t)$ contains the term $t^ke^{\lambda t}$, then $y_p$ contains the expression

$$
\sum_{j=0}^{k} A_j t^j e^{\lambda t} = A_k t^k e^{\lambda t} + \ldots + A_1 t e^{\lambda t} + A_0 e^{\lambda t}.
$$

2. If $f(t)$ contains either the term $t^k e^{at} \sin(bt)$ or the term $t^k e^{at} \cos(bt)$, but not necessarily both, then $y_p$ contains the expression

$$
\sum_{j=0}^{k} A_j t^j e^{at} \sin(bt) + \sum_{j=0}^{k} B_j t^j e^{at} \cos(bt).
$$

3. If either of the expressions above repeats part of the homogeneous solution, then it needs to be multiplied by $t$ repeatedly until it no longer contains terms which appear in the homogeneous solution.
Let us explain the overall approach by looking at the special case

\[ y''(t) - y(t) = f(t). \]

Our initial guess for a particular solution \( y_p \) is dictated by the right hand side \( f(t) \). Some typical choices appear in the table below.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( y_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t^2 e^{2t} )</td>
<td>( A t^2 e^{2t} + B t e^{2t} + C e^{2t} )</td>
</tr>
<tr>
<td>( t e^{2t} - e^{3t} )</td>
<td>( A t e^{2t} + B e^{2t} + C e^{3t} )</td>
</tr>
<tr>
<td>( t^3 + 1 )</td>
<td>( A t^3 + B t^2 + C t + D )</td>
</tr>
<tr>
<td>( t + \cos t )</td>
<td>( A t + B + C \sin t + D \cos t )</td>
</tr>
</tbody>
</table>

These choices are dictated by rules 1 and 2. According to the last rule, we also need to adjust our initial choice whenever it repeats part of the homogeneous solution. In this case, we have \( y_h = c_1 e^t + c_2 e^{-t} \), so there is no overlap with \( y_p \) and thus no need for adjustments.
We use undetermined coefficients in order to solve the equation

\[ y''(t) - 3y'(t) + 2y(t) = 2t + 5. \]

We have \( y = y_h + y_p \) and the homogeneous solution is given by

\[ \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda - 2) = 0 \]

\[ \Rightarrow \quad y_h = c_1e^t + c_2e^{2t}. \]

To find a particular solution \( y_p \), we let \( y_p = At + B \). This gives

\[ y_p'' - 3y_p' + 2y_p = -3A + 2At + 2B, \]

so we need to have \( 2A = 2 \) and \( 2B - 3A = 5 \). It easily follows that

\[ A = 1 \quad \Rightarrow \quad B = 4 \quad \Rightarrow \quad y = c_1e^t + c_2e^{2t} + t + 4. \]
Undetermined coefficients: Example 2

- We use undetermined coefficients in order to solve the equation

$$y''(t) + 5y'(t) + 6y(t) = 8e^{2t}.$$ 

- Once again, $y = y_h + y_p$ and the homogeneous solution is given by

$$\lambda^2 + 5\lambda + 6 = 0 \implies (\lambda + 2)(\lambda + 3) = 0$$

$$\implies y_h = c_1e^{-2t} + c_2e^{-3t}.$$ 

- To find a particular solution $y_p$, we let $y_p = Ae^{2t}$. This gives

$$y_p'' + 5y_p' + 6y_p = 4Ae^{2t} + 5(2Ae^{2t}) + 6Ae^{2t} = 20Ae^{2t},$$ 

so we need to have $20A = 8$. In other words, $A = 2/5$ and thus

$$y = y_h + y_p = c_1e^{-2t} + c_2e^{-3t} + \frac{2}{5}e^{2t}.$$
We use undetermined coefficients in order to solve the equation
\[ y''(t) + 5y'(t) + 6y(t) = \sin t. \]

As in the previous example, the homogeneous solution is given by
\[ y_h = c_1 e^{-2t} + c_2 e^{-3t}. \]

To find a particular solution, we let \( y_p = A \sin t + B \cos t \) and we note that \( y'_p = A \cos t - B \sin t \), while \( y''_p = -A \sin t - B \cos t \). This gives
\[ y''_p + 5y'_p + 6y_p = 5(A - B) \sin t + 5(A + B) \cos t, \]
so we need to have \( A - B = 1/5 \) and \( A + B = 0 \).

Solving these two equations, we get \( A = 1/10 \) and \( B = -1/10 \), so
\[ y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10} \sin t - \frac{1}{10} \cos t. \]
We use undetermined coefficients in order to solve the equation

\[ y''(t) + y(t) = 2 \sin t + 4e^t. \]

The homogeneous solution \( y_h \) can be found by noting that

\[ \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i \quad \Rightarrow \quad y_h = c_1 \sin t + c_2 \cos t. \]

Let us now worry about the particular solution \( y_p \). Based on the right hand side of the given equation, a natural guess for \( y_p \) would be

\[ y_p = A \sin t + B \cos t + Ce^t. \]

However, this function repeats terms that are already present in \( y_h \), so we need to adjust these terms and seek a solution of the form

\[ y_p = At \sin t + Bt \cos t + Ce^t. \]
Differentiating the last equation twice, one finds that

\[ y_p = At \sin t + Bt \cos t + Ce^t, \]
\[ y'_p = A \sin t + At \cos t + B \cos t - Bt \sin t + Ce^t, \]
\[ y''_p = 2A \cos t - At \sin t - 2B \sin t - Bt \cos t + Ce^t. \]

We need to ensure that \( y''_p + y_p = 2 \sin t + 4e^t \) and we also have

\[ y''_p + y_p = 2A \cos t - 2B \sin t + 2Ce^t \]

by above. Comparing these two equations, we arrive at the system

\[ 2A = 0, \quad -2B = 2, \quad 2C = 4. \]

This determines the coefficients \( A, B \) and \( C \), so the solution is

\[ y = y_h + y_p = c_1 \sin t + c_2 \cos t - t \cos t + 2e^t. \]
We use undetermined coefficients in order to solve the equation

\[ y''(t) - 2y'(t) + y(t) = 2e^t + 3t + 4. \]

The homogeneous solution \( y_h \) can be found by noting that

\[ \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = 0 \quad \Rightarrow \quad y_h = c_1 e^t + c_2 te^t. \]

Next, we turn to the particular solution \( y_p \). Our initial guess

\[ y_p = Ae^t + Bt + C \]

repeats part of the homogeneous solution, so this part needs to be adjusted. Since \( te^t \) is also repeating part of \( y_h \), one needs to take

\[ y_p = At^2e^t + Bt + C. \]
Differentiating the last equation twice, one easily finds that

\[ y_p = At^2e^t + Bt + C, \]
\[ y_p' = 2Ate^t + At^2e^t + B, \]
\[ y_p'' = 2Ae^t + 4Ate^t + At^2e^t, \]
\[ y_p'' - 2y_p' + y_p = 2Ae^t + Bt + C - 2B. \]

On the other hand, we need to ensure that the solution \( y_p \) satisfies

\[ y_p'' - 2y_p' + y_p = 2e^t + 3t + 4. \]

Comparing these two expressions, we arrive at the system

\[ 2A = 2, \quad B = 3, \quad C - 2B = 4. \]

This determines the coefficients \( A, B \) and \( C \), so the solution is

\[ y = y_h + y_p = c_1e^t + c_2te^t + t^2e^t + 3t + 10. \]
Linear independence and Wronskian

**Definition 2.14 – Wronskian**

The Wronskian of the functions $y_1(t), y_2(t), \ldots, y_n(t)$ is defined as

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'_1(t) & y'_2(t) & \cdots & y'_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ y^{(n-1)}_1(t) & y^{(n-1)}_2(t) & \cdots & y^{(n-1)}_n(t) \end{bmatrix}.$$ 

**Theorem 2.15 – Linear independence and Wronskian**

Suppose that the Wronskian of some scalar functions is not identically zero. Then these scalar functions are linearly independent.

- The converse of this theorem is not true in general. For instance, the Wronskian of the functions $y_1(t) = t^2$ and $y_2(t) = t|t|$ is identically zero, but these functions are linearly independent.
Consider the general scalar linear inhomogeneous equation

\[ a_n(t)y^{(n)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = f(t). \]  

(LIE)

Suppose that \( y_1(t), y_2(t), \ldots, y_n(t) \) are linearly independent solutions of the associated homogeneous equation. A particular solution of (LIE) is then \( y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t) + \ldots + c_n(t)y_n(t) \), where the coefficients \( c_k(t) \) are determined using the equation

\[
\begin{bmatrix}
  c_1'(t) \\
  c_2'(t) \\
  \vdots \\
  c_n'(t)
\end{bmatrix} =
\begin{bmatrix}
  y_1(t) & y_2(t) & \ldots & y_n(t) \\
  y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t)
\end{bmatrix}^{-1}
\begin{bmatrix}
  0 \\
  \vdots \\
  0 \\
  f(t)/a_n(t)
\end{bmatrix}.
\]
**Theorem 2.17 – Variation of parameters (Second-order case)**

Suppose that \( y_1(t), y_2(t) \) are linearly independent solutions of

\[
a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \tag{LHE}
\]

and consider the corresponding inhomogeneous equation

\[
a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t). \tag{LIE}
\]

A particular solution of (LIE) is then provided by the formula

\[
y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{a(t)W(t)} \, dt + y_2(t) \int \frac{y_1(t)f(t)}{a(t)W(t)} \, dt,
\]

where \( W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \) is the Wronskian of \( y_1 \) and \( y_2 \).
We use variation of parameters to find a particular solution of

\[ y''(t) + y(t) = \sec t. \]

The solution of the associated homogeneous equation is given by

\[ \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i \quad \Rightarrow \quad y_h = c_1 \sin t + c_2 \cos t. \]

Letting \( y_1(t) = \sin t \) and \( y_2(t) = \cos t \), we now find that

\[ W(t) = \det \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} = -\sin^2 t - \cos^2 t = -1. \]

According to the previous theorem, a particular solution is thus

\[ y_p(t) = \sin t \int \cos t \cdot \sec t \, dt - \cos t \int \sin t \cdot \sec t \, dt \]

\[ = \sin t \int \frac{\cos t}{\cos t} \, dt - \cos t \int \frac{\sin t}{\cos t} \, dt \]

\[ = t \sin t + (\cos t) \log(\cos t). \]
Reduction of order

- Suppose that we know one solution $y_1$ of the homogeneous equation

$$a_n(t)y^{(n)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = 0 \quad \text{(LHE)}$$

and that we need to solve the associated inhomogeneous equation

$$a_n(t)z^{(n)}(t) + \ldots + a_1(t)z'(t) + a_0(t)z(t) = f(t). \quad \text{(LIE)}$$

- Then the substitution $z = y_1v$ gives rise to an equation for $v$ which involves the derivatives of $v$ but not $v$ itself. Such an equation is a lower-order equation for $v'$, so it is generally easier to solve.

- This approach can be used for any linear inhomogeneous equation. In particular, we are not assuming that the coefficients $a_k$ are constant.

- When it comes to second-order equations, one may use this approach to find all solutions of (LIE), if just one solution of (LHE) is known.
It is easy to check that \( y_1(t) = t^2 \) satisfies the homogeneous equation
\[
t^2 y''(t) - 2ty'(t) + 2y(t) = 0.
\]

We now use this fact to solve the inhomogeneous equation
\[
t^2 z''(t) - 2tz'(t) + 2z(t) = t\sqrt{t}, \quad t > 0.
\]

First of all, we change variables by letting \( z = y_1 v \). This gives
\[
z = t^2 v, \quad z' = 2tv + t^2 v', \quad z'' = 2v + 4tv' + t^2 v''
\]
and the inhomogeneous equation that needs to be solved becomes
\[
t\sqrt{t} = t^2 z'' - 2tz' + 2z
\]
\[
= 2t^2 v + 4t^3 v' + t^4 v'' - 4t^2 v - 2t^3 v' + 2t^2 v
\]
\[
= t^4 v'' + 2t^3 v'.
\]
Setting $w = v'$ for convenience, we now arrive at the equation

$$t^4w' + 2t^3w = t\sqrt{t} \implies w' + 2t^{-1}w = t^{-5/2}.$$ 

This is a first-order linear equation with integrating factor

$$\mu = \exp \left( \int 2t^{-1} \, dt \right) = e^{2\log t + C} = K t^2.$$ 

Letting $K = 1$ for simplicity, we may finally conclude that

$$(\mu w)' = t^{-1/2} \implies \mu w = 2t^{1/2} + c_1 \implies w = 2t^{-3/2} + c_1 t^{-2}.$$ 

Since $v' = w$ and $z = t^2v$ by above, this also implies that

$$v = -4t^{-1/2} - c_1 t^{-1} + c_2 \implies z = -4t\sqrt{t} - c_1 t + c_2 t^2.$$
Summary of available methods

- **Homogeneous systems**: \( y'(t) = A(t)y(t) \).
  - Eigenvector method: if \( A(t) \) is constant and diagonalisable.
  - Matrix exponential: if \( A(t) \) is constant.
  - Solvable equations: if \( A(t) \) is lower/upper triangular.

- **Inhomogeneous systems**: \( y'(t) = A(t)y(t) + b(t) \).
  - Variation of parameters: this method applies in all cases.

- **Homogeneous scalar equations**: \( \sum_{k=0}^{n} a_k(t)y^{(k)}(t) = 0 \).
  - Characteristic equation: if the coefficients \( a_k \) are constant.
  - Reduction of order: if one solution is already known.

- **Inhomogeneous scalar equations**: \( \sum_{k=0}^{n} a_k(t)y^{(k)}(t) = f(t) \).
  - Undetermined coefficients: if \( a_k \) are constant and \( f \) is simple.
  - Variation of parameters: this method applies in all cases.
  - Reduction of order: if one homogeneous solution is known.