Chapter 2. Linear systems

Lecture notes for MA2327

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Definition 2.1 – Linear homogeneous systems

A linear homogeneous system is a system that has the form

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t), \tag{LHS}$$

where $\boldsymbol{y}(t)$ is the vector of unknowns and $A(t)$ is a square matrix.

Theorem 2.2 – Superposition principle

The set of solutions of a linear homogeneous system is closed under addition and scalar multiplication. In other words, the sum of any two solutions is a solution and scalar multiples of solutions are solutions.

- The superposition principle asserts that the solutions of (LHS) form a vector space. If one can find some solutions that form a basis for this vector space, then every solution will be a linear combination of them.

Linear independence of functions

Definition 2.3 – Linear independence of functions

The functions $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)$ are called linearly dependent, if there exist constants $c_1, c_2, \ldots, c_n$ which are not all zero such that

$$c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t) + \ldots + c_n\boldsymbol{y}_n(t) = 0 \quad \text{for all } t.$$

Otherwise, we say that the functions are linearly independent.

- Linear independence of vector-valued functions is a bit more subtle than linear independence of constant vectors. This is because the coefficients c_k are not allowed to depend on t. For instance,

$$\boldsymbol{y}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{y}_2(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

are linearly independent functions, even though one has $\boldsymbol{y}_2 = t\boldsymbol{y}_1$.

Linear independence of functions: Example

- We check that $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \boldsymbol{y}_3(t)$ are linearly independent when

$$\boldsymbol{y}_1(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}, \qquad \boldsymbol{y}_2(t) = \begin{bmatrix} t \\ t \end{bmatrix}, \qquad \boldsymbol{y}_3(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- Suppose that $c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t) + c_3\boldsymbol{y}_3(t) = 0$, in which case

$$c_1 e^t + c_2 t + c_3 = c_1 t + c_2 t + c_3 = 0 \quad \text{for all } t.$$

  One may analyse this relation by considering special values of t or by differentiating, for instance. Differentiating twice, we get $c_1 e^t = 0$ for all t, hence also $c_1 = 0$. The given relation may thus be reduced to

$$c_2 t + c_3 = 0 \quad \text{for all } t.$$

- Letting $t = 0$ and $t = 1$, we now get $c_3 = 0 = c_2 + c_3$. This implies that $c_k = 0$ for all k, so the given functions are linearly independent.

Linear independence of solutions

Theorem 2.4 – Linear independence of solutions

Suppose that $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)$ are solutions of the $n \times n$ system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t). \tag{LHS}$$

Then $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)$ are linearly independent functions if and only if $\boldsymbol{y}_1(0), \boldsymbol{y}_2(0), \ldots, \boldsymbol{y}_n(0)$ are linearly independent vectors.

- The solutions of an $n \times n$ linear homogeneous system form a vector space of dimension n. In fact, let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be any basis of $\mathbb{R}^n$ and let $\boldsymbol{y}_k(t)$ be the unique solution of the initial value problem

$$\boldsymbol{y}_k'(t) = A(t)\boldsymbol{y}_k(t), \qquad \boldsymbol{y}_k(0) = \boldsymbol{v}_k.$$

- Then $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)$ are easily seen to form a basis for the space of solutions. However, such a basis is not usually explicit.

Basis of solutions: Example, page 1

- We obtain a basis of solutions for the linear homogeneous system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t), \qquad A(t) = \begin{bmatrix} 1 & 0 \\ e^t & 2 \end{bmatrix}.$$

- In this case, $A(t)$ is lower triangular, so it is easier to look at the corresponding equations one by one. Let us start by writing

$$x'(t) = x(t), \qquad y'(t) = e^t x(t) + 2y(t).$$

- When it comes to the leftmost equation, one clearly has

$$x'(t) = x(t) \quad \Longrightarrow \quad x(t) = c_1 e^t.$$

- We now insert this fact in the rightmost equation to find that

$$y'(t) - 2y(t) = e^t x(t) = c_1 e^{2t}.$$

This is a first-order linear equation with integrating factor $\mu = e^{-2t}$.

Basis of solutions: Example, page 2

- Multiplying by the integrating factor, we conclude that

$$\begin{aligned}(e^{-2t}y)' = c_1 \quad &\Longrightarrow \quad e^{-2t}y(t) = c_1 t + c_2 \\ &\Longrightarrow \quad y(t) = (c_1 t + c_2)e^{2t}.\end{aligned}$$

- This shows that every solution of the system has the form

$$\boldsymbol{y}(t) = \begin{bmatrix} c_1 e^t \\ (c_1 t + c_2)e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}.$$

- In other words, every solution is a linear combination of

$$\boldsymbol{y}_1(t) = \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix}, \qquad \boldsymbol{y}_2(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$$

and these functions form a basis for the space of solutions.

- When A is constant, the linear system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$ can always be solved explicitly by relating A to its Jordan form, say $J = B^{-1}AB$.
- More precisely, the change of variables $\boldsymbol{z}(t) = B^{-1}\boldsymbol{y}(t)$ gives

$$\boldsymbol{z}'(t) = B^{-1}\boldsymbol{y}'(t) = B^{-1}A\boldsymbol{y}(t) = B^{-1}AB\boldsymbol{z}(t) = J\boldsymbol{z}(t).$$

- This is a linear system that involves a lower triangular matrix, while each of the corresponding equations has the form

$$z_k' = \lambda_k z_k \quad \text{or} \quad z_k' = z_{k-1} + \lambda_k z_k.$$

  In particular, each of these equations is first-order linear and one may determine the variables z_k inductively using integrating factors.
- The corresponding formula for the solution $\boldsymbol{y}(t) = B\boldsymbol{z}(t)$ turns out to be simple when A is diagonalisable but a bit technical, otherwise. We shall thus use another approach to deal with the general case later.

Eigenvector method

Theorem 2.5 – Eigenvector method

Consider the $n \times n$ linear system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$ in the case that A is constant and diagonalisable. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be linearly independent eigenvectors and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues. Then every solution of the system has the form

$$\boldsymbol{y}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \ldots + c_n e^{\lambda_n t}\boldsymbol{v}_n.$$

- The coefficients c_k may be taken to be real, if the eigenvalues of A are all real. Otherwise, the coefficients c_k will generally be complex.
- To prove this theorem, we note that each $\boldsymbol{y}_k(t) = e^{\lambda_k t}\boldsymbol{v}_k$ satisfies

$$\boldsymbol{y}_k'(t) = \lambda_k e^{\lambda_k t}\boldsymbol{v}_k = e^{\lambda_k t} A\boldsymbol{v}_k = A\boldsymbol{y}_k(t).$$

  This gives n solutions which are linearly independent when $t = 0$, so every other solution must be a linear combination of them.

Eigenvector method: Example 1

- We use the eigenvector method to solve the linear system

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}.$$

- The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1),$$

namely $\lambda_1 = 7$ and $\lambda_2 = 1$. These correspond to the eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- According to the previous theorem, the solution of the system is

$$\boldsymbol{y}(t) = c_1 e^{7t}\boldsymbol{v}_1 + c_2 e^{t}\boldsymbol{v}_2 = \begin{bmatrix} c_1 e^{7t} + c_2 e^{t} \\ 2c_1 e^{7t} - c_2 e^{t} \end{bmatrix}.$$

Eigenvector method: Example 2

- We use the eigenvector method to solve the linear system

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}.$$

- Since A is lower triangular, its eigenvalues $\lambda = 1, 3, 4$ are merely the diagonal entries of A. These are distinct, so A is diagonalisable and one may easily check that the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- In view of the previous theorem, the solution of the system is thus

$$\boldsymbol{y}(t) = c_1 e^t \boldsymbol{v}_1 + c_2 e^{3t} \boldsymbol{v}_2 + c_3 e^{4t} \boldsymbol{v}_3 = \begin{bmatrix} -3c_1 e^t \\ 3c_1 e^t - c_2 e^{3t} \\ c_1 e^t + c_2 e^{3t} + c_3 e^{4t} \end{bmatrix}.$$

Matrix exponential: Definition

Definition 2.6 – Matrix exponential

Given a square matrix A, we define its exponential e^A as the series

$$e^A = I + A + \frac{1}{2!}A^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

It can be shown that this series converges for every square matrix A.

- To compute the powers of a square matrix, one relates them to the powers of its Jordan form $J = B^{-1}AB$ using the computation

$$A^k = (BJB^{-1})^k = BJ^kB^{-1}.$$

- A similar approach can be used for the exponential of A since

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = \sum_{k=0}^{\infty} \frac{1}{k!}BJ^kB^{-1} = Be^JB^{-1}.$$

Matrix exponential: Properties

Theorem 2.7 – Properties of the matrix exponential

Suppose A, B are $n \times n$ matrices and let $\Phi(t) = e^{tA}$ for all $t \in \mathbb{R}$.

1. The exponential property $e^{A+B} = e^A e^B$ holds when $AB = BA$, but this property is generally false for arbitrary matrices.
2. The exponential function $\Phi(t) = e^{tA}$ is such that $\Phi'(t) = A\Phi(t)$. In particular, it is a matrix solution of the system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$.
3. The columns of $\Phi(t) = e^{tA}$ are vector solutions of $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$ and they also form a basis for the space of all solutions.

- The second property may be stated simply as $(e^{tA})' = Ae^{tA}$. This resembles the chain rule for the standard exponential function.
- The product rule $(AB)' = A'B + AB'$ also holds for matrix-valued functions, but the chain rule $(A^2)' = 2AA'$ is generally false.

Matrix exponential: Jordan forms

Theorem 2.8 – Matrix exponential of a Jordan form

Suppose that J is a $k \times k$ Jordan block with eigenvalue λ. Then the exponential e^{tJ} is a lower triangular matrix and the entries that lie i steps below the diagonal are equal to $\frac{t^j}{j!}e^{\lambda t}$ for each $0 \leq j < k$.

- For instance, the exponential of a 3×3 Jordan block is given by

$$J = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix} \quad \Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{\lambda t} & & \\ te^{\lambda t} & e^{\lambda t} & \\ \frac{t^2}{2}e^{\lambda t} & te^{\lambda t} & e^{\lambda t} \end{bmatrix}.$$

- The exponential of a Jordan form is obtained by exponentiating each Jordan block separately. As a typical example, one has

$$J = \left[\begin{array}{c|cc} 2 & & \\ \hline & 3 & \\ & 1 & 3 \end{array}\right] \quad \Longrightarrow \quad e^{tJ} = \left[\begin{array}{c|cc} e^{2t} & & \\ \hline & e^{3t} & \\ & te^{3t} & e^{3t} \end{array}\right].$$

Matrix exponential: Example 1, page 1

- We compute the matrix exponential of the diagonalisable matrix

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

- The characteristic polynomial of this matrix is given by

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5),$$

  so the eigenvalues are real and distinct, namely $\lambda_1 = 2$ and $\lambda_2 = 5$.

- The corresponding eigenvectors are easily found to be

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

  Once we now merge the eigenvectors to form a matrix B, we get

$$B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 2 & \\ & 5 \end{bmatrix}.$$

Matrix exponential: Example 1, page 2

- Since the Jordan form J is diagonal, the same is true for e^{tJ} and

$$J = B^{-1}AB = \begin{bmatrix} 2 & \\ & 5 \end{bmatrix} \quad \Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{2t} & \\ & e^{5t} \end{bmatrix}.$$

- As for the exponential of the original matrix A, this is given by

$$J = B^{-1}AB \quad \Longrightarrow \quad A = BJB^{-1} \quad \Longrightarrow \quad e^{tA} = Be^{tJ}B^{-1}.$$

- In view of our computations above, we must thus have

$$\begin{aligned} e^{tA} &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{5t} \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{5t} & -e^{2t} + e^{5t} \\ -2e^{2t} + 2e^{5t} & 2e^{2t} + e^{5t} \end{bmatrix}. \end{aligned}$$

- The exact same approach applies for any diagonalisable matrix A.

Matrix exponential: Example 2, page 1

- We compute the matrix exponential of the non-diagonalisable matrix

$$A = \begin{bmatrix} 9 & -4 \\ 9 & -3 \end{bmatrix}.$$

- In this case, the characteristic polynomial of A is given by

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so the only eigenvalue is $\lambda = 3$. The only eigenvector turns out to be

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- This implies that A is not diagonalisable and that the Jordan form is

$$J = B^{-1}AB = \begin{bmatrix} 3 & \\ 1 & 3 \end{bmatrix} \quad \Longrightarrow \quad e^{tJ} = \begin{bmatrix} e^{3t} & \\ te^{3t} & e^{3t} \end{bmatrix}.$$

Let us now find a matrix B such that $J = B^{-1}AB$ is in Jordan form.

Matrix exponential: Example 2, page 2

- Pick any nonzero vector $\boldsymbol{v}_1$ which is not an eigenvector and let

$$\boldsymbol{v}_2 = (A - \lambda I)\boldsymbol{v}_1, \qquad B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}.$$

- There are obviously infinitely many choices and one possibility is

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 6 \\ 0 & 9 \end{bmatrix}.$$

- In view of our computations above, we must thus have

$$\begin{aligned} e^{tA} = Be^{tJ}B^{-1} &= \begin{bmatrix} 1 & 6 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} e^{3t} & \\ te^{3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -2/3 \\ 0 & 1/9 \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} 1 + 6t & -4t \\ 9t & 1 - 6t \end{bmatrix}. \end{aligned}$$

- This approach applies for any non-diagonalisable 2×2 matrix A.

- Finally, we consider a real matrix A with complex eigenvalues, say

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

- The characteristic polynomial of this matrix is easily found to be

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1.$$

  The eigenvalues $\lambda = 1 \pm i$ are complex conjugates of one another and the same is true for the corresponding eigenvectors which are given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

- This implies that A is diagonalisable and that we also have

$$B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 1+i & \\ & 1-i \end{bmatrix}.$$

  In particular, one may proceed as before to compute e^{tJ} and then e^{tA}.

Matrix exponential: Example 3, page 2

- Since the Jordan form J is diagonal, the same is true for e^{tJ} and

$$J = B^{-1}AB = \begin{bmatrix} 1+i & \\ & 1-i \end{bmatrix} \implies e^{tJ} = \begin{bmatrix} e^t e^{it} & \\ & e^t e^{-it} \end{bmatrix}.$$

- In view of our computations above, we must thus have

$$\begin{aligned} e^{tA} = Be^{tJ}B^{-1} &= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^t e^{it} & \\ & e^t e^{-it} \end{bmatrix} \begin{bmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{bmatrix} \\ &= \frac{e^t}{2} \begin{bmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ i(e^{-it} - e^{it}) & e^{it} + e^{-it} \end{bmatrix}. \end{aligned}$$

- On the other hand, one has $e^{\pm it} = \cos t \pm i \sin t$, so this implies that

$$e^{tA} = \frac{e^t}{2} \begin{bmatrix} 2\cos t & -2\sin t \\ 2\sin t & 2\cos t \end{bmatrix} = \begin{bmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{bmatrix}.$$

Needless to say, e^{tA} will always turn out to be real when A is real.

Fundamental matrix

Definition 2.9 – Fundamental matrix

We say that $\Phi(t)$ is a fundamental matrix for a linear homogeneous system, if the columns of $\Phi(t)$ form a basis for the space of solutions.

- The most common example is the matrix exponential $\Phi(t) = e^{tA}$. It is a fundamental matrix for the system $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$, if A is constant.

Theorem 2.10 – Properties of fundamental matrices

Let $\Phi(t)$ be a fundamental matrix for the system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$.

1. Every solution is a linear combination of the columns of $\Phi(t)$.
2. Every solution has the form $\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c}$ for some vector $\boldsymbol{c}$.
3. The fundamental matrix itself is a matrix solution of the system. In other words, one has the matrix identity $\Phi'(t) = A(t)\Phi(t)$.

Fundamental matrix: Special cases

- It is only in a few special cases that one may explicitly determine a fundamental matrix for the linear system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$.
- When $A(t)$ is either upper or lower triangular, the system can be solved explicitly by solving the corresponding equations one by one. Let $\boldsymbol{y}_k(t)$ be the unique solution of the initial value problem

$$\boldsymbol{y}_k'(t) = A(t)\boldsymbol{y}_k(t), \qquad \boldsymbol{y}_k(0) = \boldsymbol{e}_k.$$

  Then $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)$ form a basis for the space of solutions.
- When $A(t)$ is a matrix that commutes with its antiderivative $B(t)$, a fundamental matrix for the system is given by

$$\Phi(t) = e^{B(t)}, \qquad B(t) = \int_0^t A(s)\, ds.$$

  This is the case, in particular, when $A(t) = A$ is a constant matrix.

Variation of parameters: Intuition

- Let us now turn our attention to the linear inhomogeneous system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{b}(t). \tag{LIS}$$

  When it comes to the special case $\boldsymbol{b}(t) = 0$, there is a fundamental matrix $\Phi(t)$ which satisfies the identity $\Phi'(t) = A(t)\Phi(t)$ and every solution has the form $\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c}$ for some constant vector $\boldsymbol{c}$.

- To deal with the general case, we look for solutions that have the form $\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c}(t)$, where $\boldsymbol{c}(t)$ is not necessarily constant. Since

$$\begin{aligned} \boldsymbol{y}'(t) &= \Phi'(t)\boldsymbol{c}(t) + \Phi(t)\boldsymbol{c}'(t) \\ &= A(t)\Phi(t)\boldsymbol{c}(t) + \Phi(t)\boldsymbol{c}'(t) \\ &= A(t)\boldsymbol{y}(t) + \Phi(t)\boldsymbol{c}'(t), \end{aligned}$$

  we do obtain a solution of (LIS), provided that $\Phi(t)\boldsymbol{c}'(t) = \boldsymbol{b}(t)$.

- Thus, one may use $\Phi(t)$ to solve the inhomogeneous system as well.

Variation of parameters: Main result

Theorem 2.11 – Variation of parameters

Consider the linear inhomogeneous system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{b}(t). \tag{LIS}$$

If $A(t)$ and $\boldsymbol{b}(t)$ are continuous, then every solution has the form

$$\boldsymbol{y}(t) = \Phi(t)\boldsymbol{c} + \Phi(t)\int \Phi(t)^{-1}\boldsymbol{b}(t)\,dt,$$

where $\boldsymbol{c}$ is a constant vector and $\Phi(t)$ is a fundamental matrix for the associated linear homogeneous system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$.

- The integral term in the equation above is itself a particular solution of the system. According to the theorem, every solution is thus the sum of the homogeneous solution $\Phi(t)\boldsymbol{c}$ and a particular solution.

Variation of parameters: Example

- We use variation of parameters to solve the inhomogeneous system

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \boldsymbol{b}(t), \qquad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{b}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

- Since A is constant, a fundamental matrix is $\Phi(t) = e^{tA}$ and thus

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{tA}\boldsymbol{c} + e^{tA}\int e^{-tA}\boldsymbol{b}(t)\,dt \\
&= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\int \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix} dt \\
&= \begin{bmatrix} c_1 e^t \\ c_1 te^t + c_2 e^t \end{bmatrix} + \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\int \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} c_1 e^t \\ c_1 te^t + c_2 e^t \end{bmatrix} + \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^t - 1 \\ c_1 te^t + c_2 e^t + te^t - t \end{bmatrix}.
\end{aligned}$$

Higher-order scalar equations

- Suppose that we need to solve a scalar linear equation such as

$$y'''(t) - 5y''(t) + 7y'(t) - 3y(t) = 0.$$

This is a 3rd-order equation, so one may express it as a 3×3 system.

- More precisely, let $\boldsymbol{y}$ be the vector with entries y, y', y'' and note that

$$\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = A\boldsymbol{y}.$$

- Since the scalar equation is linear, the same is true for the system, so one may determine $\boldsymbol{y}$ using methods we have already developed.
- This kind of approach is certainly valid, but it is not very efficient, as we are only interested in the first entry of $\boldsymbol{y}$. It is thus worth having some related results that deal with scalar equations directly.

Linear homogeneous equations

Theorem 2.12 – Linear homogeneous equations

Consider the scalar linear homogeneous equation

$$a_n y^{(n)}(t) + \ldots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0. \qquad \text{(LHE)}$$

If the coefficients a_k are all constant, then one may obtain a basis of solutions by solving the corresponding characteristic equation

$$a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

and by associating each root λ with solutions of (LHE) as follows.

1. If a real root λ has multiplicity k, it gets associated with the k functions $\{t^j e^{\lambda t}\}_{j=0}^{k-1}$, namely with $e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1} e^{\lambda t}$.
2. If a pair of complex roots $\lambda = a \pm bi$ has multiplicity k, it gets associated with the $2k$ functions $\{t^j e^{at} \sin(bt), t^j e^{at} \cos(bt)\}_{j=0}^{k-1}$.

Linear homogeneous equations: Example 1

- We use the previous theorem to solve the homogeneous equation

$$y'''(t) - 5y''(t) + 7y'(t) - 3y(t) = 0.$$

- In this case, the associated characteristic equation is given by

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

- Noting that $\lambda = 1$ is a root, one easily finds that

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = (\lambda - 1)(\lambda^2 - 4\lambda + 3) = (\lambda - 1)^2(\lambda - 3).$$

- This means that $\lambda = 1$ is a double root, while $\lambda = 3$ is a simple root. Thus, a basis of solutions is formed by the functions e^t, te^t, e^{3t} and every solution of the given equation has the form

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{3t}.$$

Linear homogeneous equations: Example 2

- Let us now solve an initial value problem such as

$$y''(t) - y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 3.$$

- In this case, the associated characteristic equation gives

$$\lambda^2 - 1 = 0 \implies (\lambda+1)(\lambda-1) = 0 \implies \lambda = -1, 1.$$

  Since the roots are both simple, every solution has the form

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

- Next, we turn to the initial conditions and we note that

$$\begin{aligned} y(t) = c_1 e^t + c_2 e^{-t} &\implies 1 = y(0) = c_1 + c_2, \\ y'(t) = c_1 e^t - c_2 e^{-t} &\implies 3 = y'(0) = c_1 - c_2. \end{aligned}$$

- Solving this system of equations, we find that $c_1 = 2$ and $c_2 = -1$. Thus, the unique solution is given by $y(t) = 2e^t - e^{-t}$.

Linear homogeneous equations: Example 3

- The equation that describes a simple harmonic oscillator is

$$my''(t) = -ky(t).$$

  Here, the constants k, m are both positive, so one may also write

$$y''(t) + \omega^2 y(t) = 0, \qquad \omega = \sqrt{k/m}.$$

- Solving the associated characteristic equation, we now get

$$\lambda^2 + \omega^2 = 0 \quad \Longrightarrow \quad \lambda^2 = -\omega^2 \quad \Longrightarrow \quad \lambda = \pm i\omega.$$

  This is a pair of complex roots, so every solution has the form

$$y(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t).$$

- In particular, every solution of the given equation is periodic.

Linear homogeneous equations: Example 4

- As our last example on homogeneous equations, we now solve

$$y'''(t) + 7y''(t) + 19y'(t) + 13y(t) = 0.$$

- In this case, the associated characteristic equation is given by

$$\lambda^3 + 7\lambda^2 + 19\lambda + 13 = 0.$$

  Noting that $\lambda = -1$ is a root, one may factor the cubic as

$$\lambda^3 + 7\lambda^2 + 19\lambda + 13 = (\lambda + 1)(\lambda^2 + 6\lambda + 13).$$

- The roots of the quadratic factor are easily found to be

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 13}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$

- We may thus conclude that every solution has the form

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} \sin(2t) + c_3 e^{-3t} \cos(2t).$$

Linear inhomogeneous equations

- Suppose that we need to solve an inhomogeneous equation such as

$$y''(t) - 3y'(t) + 2y(t) = 2t + 5.$$

- The solution of such an equation can be expressed as the sum of the homogeneous solution y_h and a particular solution y_p. More precisely, the difference $z = y - y_p$ between any two solutions satisfies

$$z''(t) - 3z'(t) + 2z(t) = 0,$$

  so it is a solution of the corresponding homogeneous equation.
- This proves the useful identity $y = y_h + y_p$. We already know how to find the homogeneous solution y_h, so we need only worry about y_p.
- There are two methods for finding a particular solution: the method of undetermined coefficients and variation of parameters. The former is generally simpler, but it only applies in a few special cases.

Undetermined coefficients: Main result

Theorem 2.13 – Undetermined coefficients

Consider the scalar linear inhomogeneous equation

$$a_n y^{(n)}(t) + \ldots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t). \qquad \text{(LIE)}$$

Suppose that the coefficients a_k are all constant and that the right hand side $f(t)$ is a linear combination of terms that have the form

$$t^j e^{\lambda t}, \qquad t^j e^{at}\sin(bt), \qquad t^j e^{at}\cos(bt).$$

Then the solution $y(t)$ satisfies a higher-order homogeneous equation, so it can itself be expressed as a linear combination of such terms.

- One typically uses this theorem to write down an explicit formula for a particular solution y_p. It is easy to predict the terms that appear in the formula, but their exact coefficients need to be determined.

Undetermined coefficients: General rules

The general rules for finding a particular solution y_p are the following.

1. If $f(t)$ contains the term $t^k e^{\lambda t}$, then y_p contains the expression

$$\sum_{j=0}^{k} A_j t^j e^{\lambda t} = A_k t^k e^{\lambda t} + \ldots + A_1 t e^{\lambda t} + A_0 e^{\lambda t}.$$

2. If $f(t)$ contains either the term $t^k e^{at} \sin(bt)$ or the term $t^k e^{at} \cos(bt)$, but not necessarily both, then y_p contains the expression

$$\sum_{j=0}^{k} A_j t^j e^{at} \sin(bt) + \sum_{j=0}^{k} B_j t^j e^{at} \cos(bt).$$

3. If either of the expressions above repeats part of the homogeneous solution, then it needs to be multiplied by t repeatedly until it no longer contains terms which appear in the homogeneous solution.

Undetermined coefficients: Some comments

- Let us explain the overall approach by looking at the special case

$$y''(t) - y(t) = f(t).$$

- Our initial guess for a particular solution y_p is dictated by the right hand side $f(t)$. Some typical choices appear in the table below.

$f(t)$	y_p
t^2e^{2t}	$At^2e^{2t} + Bte^{2t} + Ce^{2t}$
$te^{2t} - e^{3t}$	$Ate^{2t} + Be^{2t} + Ce^{3t}$
$t^3 + 1$	$At^3 + Bt^2 + Ct + D$
$t + \cos t$	$At + B + C\sin t + D\cos t$

- These choices are dictated by rules 1 and 2. According to the last rule, we also need to adjust our initial choice whenever it repeats part of the homogeneous solution. In this case, we have $y_h = c_1e^t + c_2e^{-t}$, so there is no overlap with y_p and thus no need for adjustments.

Undetermined coefficients: Example 1

- We use undetermined coefficients in order to solve the equation

$$y''(t) - 3y'(t) + 2y(t) = 2t + 5.$$

- We have $y = y_h + y_p$ and the homogeneous solution is given by

$$\begin{aligned} \lambda^2 - 3\lambda + 2 = 0 \quad &\implies \quad (\lambda - 1)(\lambda - 2) = 0 \\ &\implies \quad y_h = c_1 e^t + c_2 e^{2t}. \end{aligned}$$

- To find a particular solution y_p, we let $y_p = At + B$. This gives

$$y_p'' - 3y_p' + 2y_p = -3A + 2At + 2B,$$

so we need to have $2A = 2$ and $2B - 3A = 5$. It easily follows that

$$A = 1 \quad \implies \quad B = 4 \quad \implies \quad y = c_1 e^t + c_2 e^{2t} + t + 4.$$

Undetermined coefficients: Example 2

- We use undetermined coefficients in order to solve the equation

$$y''(t) + 5y'(t) + 6y(t) = 8e^{2t}.$$

- Once again, $y = y_h + y_p$ and the homogeneous solution is given by

$$\begin{aligned} \lambda^2 + 5\lambda + 6 = 0 \quad &\Longrightarrow \quad (\lambda + 2)(\lambda + 3) = 0 \\ &\Longrightarrow \quad y_h = c_1 e^{-2t} + c_2 e^{-3t}. \end{aligned}$$

- To find a particular solution y_p, we let $y_p = Ae^{2t}$. This gives

$$y_p'' + 5y_p' + 6y_p = 4Ae^{2t} + 5(2Ae^{2t}) + 6Ae^{2t} = 20Ae^{2t},$$

so we need to have $20A = 8$. In other words, $A = 2/5$ and thus

$$y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-3t} + \frac{2}{5} e^{2t}.$$

Undetermined coefficients: Example 3

- We use undetermined coefficients in order to solve the equation

$$y''(t) + 5y'(t) + 6y(t) = \sin t.$$

- As in the previous example, the homogeneous solution is given by

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}.$$

- To find a particular solution, we let $y_p = A\sin t + B\cos t$ and we note that $y_p' = A\cos t - B\sin t$, while $y_p'' = -A\sin t - B\cos t$. This gives

$$y_p'' + 5y_p' + 6y_p = 5(A-B)\sin t + 5(A+B)\cos t,$$

  so we need to have $A - B = 1/5$ and $A + B = 0$.
- Solving these two equations, we get $A = 1/10$ and $B = -1/10$, so

$$y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10}\sin t - \frac{1}{10}\cos t.$$

- We use undetermined coefficients in order to solve the equation

$$y''(t) + y(t) = 2\sin t + 4e^t.$$

- The homogeneous solution y_h can be found by noting that

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i \quad \Longrightarrow \quad y_h = c_1 \sin t + c_2 \cos t.$$

- Let us now worry about the particular solution y_p. Based on the right hand side of the given equation, a natural guess for y_p would be

$$y_p = A\sin t + B\cos t + Ce^t.$$

- However, this function repeats terms that are already present in y_h, so we need to adjust these terms and seek a solution of the form

$$y_p = At\sin t + Bt\cos t + Ce^t.$$

Undetermined coefficients: Example 4, page 2

- Differentiating the last equation twice, one finds that

$$\begin{aligned} y_p &= At\sin t + Bt\cos t + Ce^t, \\ y_p' &= A\sin t + At\cos t + B\cos t - Bt\sin t + Ce^t, \\ y_p'' &= 2A\cos t - At\sin t - 2B\sin t - Bt\cos t + Ce^t. \end{aligned}$$

- We need to ensure that $y_p'' + y_p = 2\sin t + 4e^t$ and we also have

$$y_p'' + y_p = 2A\cos t - 2B\sin t + 2Ce^t$$

by above. Comparing these two equations, we arrive at the system

$$2A = 0, \qquad -2B = 2, \qquad 2C = 4.$$

- This determines the coefficients A, B and C, so the solution is

$$y = y_h + y_p = c_1\sin t + c_2\cos t - t\cos t + 2e^t.$$

Undetermined coefficients: Example 5, page 1

- We use undetermined coefficients in order to solve the equation

$$y''(t) - 2y'(t) + y(t) = 2e^t + 3t + 4.$$

- The homogeneous solution y_h can be found by noting that

$$\begin{aligned} \lambda^2 - 2\lambda + 1 = 0 \quad &\implies \quad (\lambda - 1)^2 = 0 \\ &\implies \quad y_h = c_1 e^t + c_2 t e^t. \end{aligned}$$

- Next, we turn to the particular solution y_p. Our initial guess

$$y_p = Ae^t + Bt + C$$

repeats part of the homogeneous solution, so this part needs to be adjusted. Since te^t is also repeating part of y_h, one needs to take

$$y_p = At^2 e^t + Bt + C.$$

Undetermined coefficients: Example 5, page 2

- Differentiating the last equation twice, one easily finds that

$$\begin{aligned} y_p &= At^2e^t + Bt + C, \\ y_p' &= 2Ate^t + At^2e^t + B, \\ y_p'' &= 2Ae^t + 4Ate^t + At^2e^t, \\ y_p'' - 2y_p' + y_p &= 2Ae^t + Bt + C - 2B. \end{aligned}$$

- On the other hand, we need to ensure that the solution y_p satisfies

$$y_p'' - 2y_p' + y_p = 2e^t + 3t + 4.$$

- Comparing these two expressions, we arrive at the system

$$2A = 2, \qquad B = 3, \qquad C - 2B = 4.$$

- This determines the coefficients A, B and C, so the solution is

$$y = y_h + y_p = c_1e^t + c_2te^t + t^2e^t + 3t + 10.$$

Linear independence and Wronskian

Definition 2.14 – Wronskian

The Wronskian of the functions $y_1(t), y_2(t), \ldots, y_n(t)$ is defined as

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{bmatrix}.$$

Theorem 2.15 – Linear independence and Wronskian

Suppose that the Wronskian of some scalar functions is not identically zero. Then these scalar functions are linearly independent.

- The converse of this theorem is not true in general. For instance, the Wronskian of the functions $y_1(t) = t^2$ and $y_2(t) = t|t|$ is identically zero, but these functions are linearly independent.

Theorem 2.16 – Variation of parameters (General case)

Consider the general scalar linear inhomogeneous equation

$$a_n(t)y^{(n)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = f(t). \tag{LIE}$$

Suppose that $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent solutions of the associated homogeneous equation. A particular solution of (LIE) is then $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t) + \ldots + c_n(t)y_n(t)$, where the coefficients $c_k(t)$ are determined using the equation

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \\ \vdots \\ c_n'(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t)/a_n(t) \end{bmatrix}.$$

Variation of parameters: Second-order case

Theorem 2.17 – Variation of parameters (Second-order case)

Suppose that $y_1(t), y_2(t)$ are linearly independent solutions of

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \tag{LHE}$$

and consider the corresponding inhomogeneous equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t). \tag{LIE}$$

A particular solution of (LIE) is then provided by the formula

$$y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{a(t)W(t)}\,dt + y_2(t) \int \frac{y_1(t)f(t)}{a(t)W(t)}\,dt,$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian of y_1 and y_2.

Variation of parameters: Example

- We use variation of parameters to find a particular solution of

$$y''(t) + y(t) = \sec t.$$

- The solution of the associated homogeneous equation is given by

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i \quad \Longrightarrow \quad y_h = c_1 \sin t + c_2 \cos t.$$

- Letting $y_1(t) = \sin t$ and $y_2(t) = \cos t$, we now find that

$$W(t) = \det \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} = -\sin^2 t - \cos^2 t = -1.$$

- According to the previous theorem, a particular solution is thus

$$\begin{aligned} y_p(t) &= \sin t \int \cos t \cdot \sec t \, dt - \cos t \int \sin t \cdot \sec t \, dt \\ &= \sin t \int \frac{\cos t}{\cos t} \, dt - \cos t \int \frac{\sin t}{\cos t} \, dt \\ &= t \sin t + (\cos t) \log(\cos t). \end{aligned}$$

Reduction of order

- Suppose that we know one solution y_1 of the homogeneous equation

$$a_n(t)y^{(n)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = 0 \tag{LHE}$$

and that we need to solve the associated inhomogeneous equation

$$a_n(t)z^{(n)}(t) + \ldots + a_1(t)z'(t) + a_0(t)z(t) = f(t). \tag{LIE}$$

- Then the substitution $z = y_1 v$ gives rise to an equation for v which involves the derivatives of v but not v itself. Such an equation is a lower-order equation for v', so it is generally easier to solve.
- This approach can be used for any linear inhomogeneous equation. In particular, we are not assuming that the coefficients a_k are constant.
- When it comes to second-order equations, one may use this approach to find all solutions of (LIE), if just one solution of (LHE) is known.

Reduction of order: Example, page 1

- It is easy to check that $y_1(t) = t^2$ satisfies the homogeneous equation

$$t^2 y''(t) - 2ty'(t) + 2y(t) = 0.$$

We now use this fact to solve the inhomogeneous equation

$$t^2 z''(t) - 2tz'(t) + 2z(t) = t\sqrt{t}, \qquad t > 0.$$

- First of all, we change variables by letting $z = y_1 v$. This gives

$$z = t^2 v, \qquad z' = 2tv + t^2 v', \qquad z'' = 2v + 4tv' + t^2 v''$$

and the inhomogeneous equation that needs to be solved becomes

$$\begin{aligned} t\sqrt{t} &= t^2 z'' - 2tz' + 2z \\ &= \cancel{2t^2 v} + 4t^3 v' + t^4 v'' - \cancel{4t^2 v} - 2t^3 v' + \cancel{2t^2 v} \\ &= t^4 v'' + 2t^3 v'. \end{aligned}$$

Reduction of order: Example, page 2

- Setting $w = v'$ for convenience, we now arrive at the equation

$$t^4 w' + 2t^3 w = t\sqrt{t} \quad \Longrightarrow \quad w' + 2t^{-1} w = t^{-5/2}.$$

This is a first-order linear equation with integrating factor

$$\mu = \exp\left(\int 2t^{-1}\,dt\right) = e^{2\log t + C} = Kt^2.$$

- Letting $K = 1$ for simplicity, we may finally conclude that

$$\begin{aligned} (\mu w)' = t^{-1/2} \quad &\Longrightarrow \quad \mu w = 2t^{1/2} + c_1 \\ &\Longrightarrow \quad w = 2t^{-3/2} + c_1 t^{-2}. \end{aligned}$$

- Since $v' = w$ and $z = t^2 v$ by above, this also implies that

$$v = -4t^{-1/2} - c_1 t^{-1} + c_2 \quad \Longrightarrow \quad z = -4t\sqrt{t} - c_1 t + c_2 t^2.$$

Summary of available methods

- **Homogeneous systems:** $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$.

  $\longrightarrow$ Eigenvector method: if $A(t)$ is constant and diagonalisable.

  $\longrightarrow$ Matrix exponential: if $A(t)$ is constant.

  $\longrightarrow$ Solvable equations: if $A(t)$ is lower/upper triangular.

- **Inhomogeneous systems:** $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$.

  $\longrightarrow$ Variation of parameters: this method applies in all cases.

- **Homogeneous scalar equations:** $\sum_{k=0}^{n} a_k(t) y^{(k)}(t) = 0$.

  $\longrightarrow$ Characteristic equation: if the coefficients a_k are constant.

  $\longrightarrow$ Reduction of order: if one solution is already known.

- **Inhomogeneous scalar equations:** $\sum_{k=0}^{n} a_k(t) y^{(k)}(t) = f(t)$.

  $\longrightarrow$ Undetermined coefficients: if a_k are constant and f is simple.

  $\longrightarrow$ Variation of parameters: this method applies in all cases.

  $\longrightarrow$ Reduction of order: if one homogeneous solution is known.