# MA2223 – Tutorial solutions
## Part 3. Normed vector spaces

**T9–1.** Let $X$ be a normed vector space and let $\boldsymbol{x}, \boldsymbol{y} \in X$. Show that

$$\big| ||\boldsymbol{x}|| - ||\boldsymbol{y}|| \big| \leq ||\boldsymbol{x} - \boldsymbol{y}||.$$

This result is a direct consequence of the triangle inequality because

$$\begin{aligned} ||\boldsymbol{x}|| \leq ||\boldsymbol{x} - \boldsymbol{y}|| + ||\boldsymbol{y}|| \quad &\implies \quad ||\boldsymbol{x}|| - ||\boldsymbol{y}|| \leq ||\boldsymbol{x} - \boldsymbol{y}||, \\ ||\boldsymbol{y}|| \leq ||\boldsymbol{y} - \boldsymbol{x}|| + ||\boldsymbol{x}|| \quad &\implies \quad ||\boldsymbol{y}|| - ||\boldsymbol{x}|| \leq ||\boldsymbol{x} - \boldsymbol{y}||. \end{aligned}$$

**T9–2.** Suppose $\boldsymbol{x}_n$ is a Cauchy sequence in a normed vector space $X$. Show that the norms $||\boldsymbol{x}_n||$ form a convergent sequence in $\mathbb{R}$.

Let $\varepsilon > 0$ be given. Since the sequence $\boldsymbol{x}_n$ is Cauchy, there exists $N > 0$ such that

$$||\boldsymbol{x}_m - \boldsymbol{x}_n|| < \varepsilon \quad \text{for all } m, n \geq N.$$

Consider the sequence of real numbers $t_n = ||\boldsymbol{x}_n||$. Using the previous problem, we get

$$|t_m - t_n| = \big| ||\boldsymbol{x}_m|| - ||\boldsymbol{x}_n|| \big| \leq ||\boldsymbol{x}_m - \boldsymbol{x}_n|| < \varepsilon$$

for all $m, n \geq N$. In particular, $t_n$ is a Cauchy sequence in $\mathbb{R}$ and thus convergent.

**T9–3.** Consider the identity function $I \colon (X, ||\cdot||_\infty) \to (X, ||\cdot||_1)$ in the case that $X = C[a, b]$. Show that $I$ is Lipschitz continuous.

Let $f, g \in C[a, b]$ be arbitrary. Using the definitions of the norms $||f||_p$, one finds that

$$\begin{aligned} ||I(f(x)) - I(g(x))||_1 = ||f(x) - g(x)||_1 &= \int_a^b |f(x) - g(x)|\, dx \\ &\leq \int_a^b ||f(x) - g(x)||_\infty\, dx \\ &= (b - a) \cdot ||f(x) - g(x)||_\infty. \end{aligned}$$

**T9–4.** Consider the space $X = C[0,1]$ and the sequence $f_n(x) = \sqrt{n} \cdot x^n$. Show that $f_n \to 0$ in $(X, ||\cdot||_1)$. Is the same true in $(X, ||\cdot||_2)$?

When it comes to the first norm, it is easy to check that

$$||f_n||_1 = \int_0^1 |f_n(x)|\, dx = \sqrt{n} \int_0^1 x^n\, dx = \frac{\sqrt{n}}{n+1}$$

and this expression obviously approaches zero as $n \to \infty$. On the other hand,

$$||f_n||_2^2 = \int_0^1 f_n(x)^2\, dx = n \int_0^1 x^{2n}\, dx = \frac{n}{2n+1}$$

and this expression approaches $\frac{1}{2}$ as $n \to \infty$, so we also have $||f_n||_2 \to \frac{1}{\sqrt{2}}$ as $n \to \infty$.

**T9–5.** Consider the unit sphere $S = \{\boldsymbol{x} \in X : ||\boldsymbol{x}|| = 1\}$. Show that $S$ is closed in every normed vector space $X$.

Let $f\colon X \to \mathbb{R}$ denote the norm function $f(x) = ||\boldsymbol{x}||$ which is known to be continuous. Since the set $A = \{1\}$ is closed in $\mathbb{R}$, its inverse image is closed in $X$. On the other hand,

$$f^{-1}(A) = \{\boldsymbol{x} \in X : f(\boldsymbol{x}) = 1\} = \{\boldsymbol{x} \in X : ||\boldsymbol{x}|| = 1\} = S.$$

**T9–6.** Is the discrete metric on $\mathbb{R}$ induced by a norm?

No. If a metric is induced by a norm, then that metric must satisfy

$$d(2\boldsymbol{x}, 2\boldsymbol{y}) = ||2\boldsymbol{x} - 2\boldsymbol{y}|| = 2||\boldsymbol{x} - \boldsymbol{y}|| = 2d(\boldsymbol{x}, \boldsymbol{y}) \text{ for all } \boldsymbol{x}, \boldsymbol{y}.$$

This is not the case for the discrete metric because $d(2\boldsymbol{x}, 2\boldsymbol{y}) = d(\boldsymbol{x}, \boldsymbol{y}) = 1$ for all $\boldsymbol{x} \neq \boldsymbol{y}$.

**T10–1.** Show that $||S \circ T|| \leq ||S|| \cdot ||T||$ whenever $S, T \in L(X, X)$.

First, we show that $||T(\boldsymbol{x})|| \leq ||T|| \cdot ||\boldsymbol{x}||$ for each $T \in L(X, X)$. When $\boldsymbol{x} \neq 0$, we have

$$||T|| = \sup_{\boldsymbol{x} \neq 0} \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||} \quad \Longrightarrow \quad \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||} \leq ||T|| \quad \Longrightarrow \quad ||T(\boldsymbol{x})|| \leq ||T|| \cdot ||\boldsymbol{x}||$$

and the inequality holds. When $\boldsymbol{x} = 0$, we have $T(\boldsymbol{x}) = 0$, so the inequality holds in that case as well. Using the inequality twice, one now finds that

$$||S(T(\boldsymbol{x}))|| \leq ||S|| \cdot ||T(\boldsymbol{x})|| \leq ||S|| \cdot ||T|| \cdot ||\boldsymbol{x}||$$

for all $\boldsymbol{x} \in X$. In view of the definition of the operator norm, this also implies that

$$||S \circ T|| = \sup_{\boldsymbol{x} \neq 0} \frac{||S(T(\boldsymbol{x}))||}{||\boldsymbol{x}||} \leq ||S|| \cdot ||T||.$$

**T10–2.** Consider the linear operator $T\colon (\mathbb{R}^2, ||\cdot||_\infty) \to (\mathbb{R}^2, ||\cdot||_\infty)$ which is defined by $T(x_1, x_2) = (2x_1 + 3x_2, 4x_2)$. Find its operator norm.

First of all, we establish an upper bound for the operator norm. The expression

$$||T(\boldsymbol{x})||_\infty = \max(|2x_1 + 3x_2|, |4x_2|)$$

is equal to either $|2x_1 + 3x_2|$ or else $|4x_2|$. In the former case, one finds that

$$||T(\boldsymbol{x})||_\infty = |2x_1 + 3x_2| \leq 2|x_1| + 3|x_2| \leq 5 \max |x_i| = 5||\boldsymbol{x}||_\infty.$$

In the latter case, a similar argument leads to

$$||T(\boldsymbol{x})||_\infty = 4|x_2| \leq 4 \max |x_i| = 4||\boldsymbol{x}||_\infty.$$

This means that $||T(\boldsymbol{x})||_\infty \leq 5||\boldsymbol{x}||_\infty$ in any case, so the operator norm satisfies

$$||T|| = \sup_{\boldsymbol{x}\neq 0} \frac{||T(\boldsymbol{x})||_\infty}{||\boldsymbol{x}||_\infty} \leq 5.$$

Next, we show that equality holds for some particular vector $\boldsymbol{x}$. For instance, take

$$\boldsymbol{x} = (1, 1), \qquad T(\boldsymbol{x}) = (5, 4), \qquad ||\boldsymbol{x}||_\infty = 1, \qquad ||T(\boldsymbol{x})||_\infty = 5.$$

This gives $\frac{||T(\boldsymbol{x})||_\infty}{||\boldsymbol{x}||_\infty} = 5$ for some vector $\boldsymbol{x}$, so $||T|| \geq 5$ and thus $||T|| = 5$ by above.

**T10–3.** Consider the linear operator $T\colon (\mathbb{R}^2, ||\cdot||_1) \to (\mathbb{R}^2, ||\cdot||_\infty)$ which is defined by $T(x_1, x_2) = (2x_1 + 3x_2, 4x_2)$. Find its operator norm.

First of all, we establish an upper bound for the operator norm. The expression

$$||T(\boldsymbol{x})||_\infty = \max(|2x_1 + 3x_2|, |4x_2|)$$

is equal to either $|2x_1 + 3x_2|$ or else $|4x_2|$. In the former case, one finds that

$$||T(\boldsymbol{x})||_\infty = |2x_1 + 3x_2| \leq 2|x_1| + 3|x_2| \leq 3|x_1| + 3|x_2| = 3||\boldsymbol{x}||_1.$$

In the latter case, a similar argument leads to

$$||T(\boldsymbol{x})||_\infty = 4|x_2| \leq 4|x_1| + 4|x_2| = 4||\boldsymbol{x}||_1.$$

This means that $||T(\boldsymbol{x})||_\infty \leq 4||\boldsymbol{x}||_1$ in any case, so the operator norm satisfies

$$||T|| = \sup_{\boldsymbol{x}\neq 0} \frac{||T(\boldsymbol{x})||_\infty}{||\boldsymbol{x}||_1} \leq 4.$$

Next, we show that equality holds for some particular vector $\boldsymbol{x}$. For instance, take

$$\boldsymbol{x} = (0, 1), \qquad T(\boldsymbol{x}) = (3, 4), \qquad ||\boldsymbol{x}||_1 = 1, \qquad ||T(\boldsymbol{x})||_\infty = 4.$$

This gives $\frac{||T(\boldsymbol{x})||_\infty}{||\boldsymbol{x}||_1} = 4$ for some vector $\boldsymbol{x}$, so $||T|| \geq 4$ and thus $||T|| = 4$ by above.

**T10–4.** Let $\boldsymbol{a} \in \ell^\infty$ and consider the linear operator $T\colon \ell^2 \to \ell^2$ which is defined by $T(x_1, x_2, \ldots) = (a_1x_1, a_2x_2, \ldots)$. Find its operator norm.

Using the definition of the norms $||\boldsymbol{x}||_p$, one finds that

$$||T(\boldsymbol{x})||_2^2 = \sum_{i=1}^\infty a_i^2 x_i^2 \le \sum_{i=1}^\infty ||\boldsymbol{a}||_\infty^2 x_i^2 = ||\boldsymbol{a}||_\infty^2 ||\boldsymbol{x}||_2^2.$$

This proves the inequality $||T(\boldsymbol{x})|| \le ||\boldsymbol{a}||_\infty ||\boldsymbol{x}||_2$ which also implies that

$$||T|| = \sup_{\boldsymbol{x}\neq 0} \frac{||T(\boldsymbol{x})||_2}{||\boldsymbol{x}||_2} \le ||\boldsymbol{a}||_\infty.$$

Next, we show that equality holds. Consider the vector $\boldsymbol{x} = \boldsymbol{e}_k$ that has 1 as its $k$th entry and all other entries equal to zero. For that particular vector, we have

$$T(\boldsymbol{x}) = a_k \boldsymbol{x} \quad \Longrightarrow \quad ||T(\boldsymbol{x})||_2 = |a_k| \cdot ||\boldsymbol{x}||_2 \quad \Longrightarrow \quad ||T|| \ge |a_k|.$$

This holds for any index $k$, so $||T|| \ge \sup_k |a_k| = ||\boldsymbol{a}||_\infty$ and thus $||T|| = ||\boldsymbol{a}||_\infty$ by above.

**T10–5.** Show that the norms $||f||_1$ and $||f||_\infty$ are not equivalent in $C[0,1]$.

If the given norms are equivalent, then there exist constants $a, b > 0$ such that

$$a||f||_1 \le ||f||_\infty \le b||f||_1 \quad \text{for all } f \in C[0,1].$$

Let us then consider the case $f(x) = x^n$ for any integer $n \ge 1$. In this case, we have

$$||f_n||_1 = \int_0^1 x^n\,dx = \frac{1}{n+1}, \qquad ||f_n||_\infty = \sup_{0\le x\le 1} x^n = 1.$$

Were the two norms equivalent, we would be able to conclude that

$$\frac{a}{n+1} \le 1 \le \frac{b}{n+1}$$

for any integer $n \ge 1$. This is not true because both fractions approach zero as $n \to \infty$.

**T10–6.** Show that the norms $||f||_1$ and $||f||_2$ are not equivalent in $C[0,1]$.

If the given norms are equivalent, then there exist constants $a, b > 0$ such that

$$a||f||_1 \le ||f||_2 \le b||f||_1 \quad \text{for all } f \in C[0,1].$$

Let us then consider the case $f(x) = x^n$ for any integer $n \geq 1$. In this case, we have

$$||f_n||_1 = \int_0^1 x^n \, dx = \frac{1}{n+1}, \qquad ||f_n||_2 = \sqrt{\int_0^1 x^{2n} \, dx} = \frac{1}{\sqrt{2n+1}}.$$

Were the two norms equivalent, we would be able to conclude that

$$\frac{a}{n+1} \leq \frac{1}{\sqrt{2n+1}} \leq \frac{b}{n+1} \quad \Longrightarrow \quad a \leq \frac{n+1}{\sqrt{2n+1}} \leq b$$

for any integer $n \geq 1$. This is not true because $\frac{n+1}{\sqrt{2n+1}}$ can be arbitrarily large.