MA2223 – Tutorial solutions
Part 2. Topological spaces

T5–1. Let T be the collection of subsets of $\mathbb{R}$ that consists of $\varnothing, \mathbb{R}$ and every interval of the form $(-\infty, a)$. Show that T is a topology on $\mathbb{R}$.

We check the properties that a topology needs to satisfy. First of all, the sets $\varnothing, \mathbb{R}$ are open by assumption. To show that unions of open sets are open, we note that

$$\bigcup_i (-\infty, a_i) = (-\infty, \sup_i a_i).$$

Given any number of intervals that have the form $(-\infty, a_i)$, their union is then an interval of the same form, so it is open as well. The same is true for finite intersections because

$$\bigcap_{i=1}^n (-\infty, a_i) = (-\infty, \min_{1\leq i\leq n} a_i).$$

T5–2. Find the closure of $(0,1) \subset \mathbb{R}$ with respect to the discrete topology, the indiscrete topology and the topology of the previous problem.

By definition, the closure of A is the smallest closed set that contains A. If we use the discrete topology, then every set is open, so every set is closed. This implies that $\overline{A} = A$. If we use the indiscrete topology, then only $\varnothing, \mathbb{R}$ are open, so only $\varnothing, \mathbb{R}$ are closed and this implies that $\overline{A} = \mathbb{R}$. As for the topology of the previous problem, the nontrivial closed sets have the form $[a, \infty)$ and the smallest one that contains $A = (0,1)$ is the set $\overline{A} = [0, \infty)$.

T5–3. Consider $\mathbb{R}^2$ with its usual topology. Find the closure, the interior and the boundary of the upper half plane $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

The upper half plane A is open, so it is equal to its own interior, namely $A^\circ = A$. The closure must contain the points which are limits of sequences in A, so the closure is

$$\overline{A} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}.$$

Finally, the boundary of A is defined as $\partial A = \overline{A} \cap \overline{X - A}$. In this case, we have

$$\begin{aligned} X - A = \{(x, y) \in \mathbb{R}^2 : y \leq 0\} \quad &\Longrightarrow \quad \overline{X - A} = \{(x, y) \in \mathbb{R}^2 : y \leq 0\} \\ &\Longrightarrow \quad \partial A = \{(x, y) \in \mathbb{R}^2 : y = 0\}. \end{aligned}$$

T5–4. Let (X, T) be a topological space and $A \subset X$. Show that A is open if and only if each $x \in A$ has a neighbourhood U such that $U \subset A$.

If the set A is open, then $A = A^\circ$ by Theorem 2.4 and one may use Theorem 2.5 to conclude that each $x \in A$ has a neighbourhood U_x that lies entirely within A. Conversely, suppose that each $x \in A$ has a neighbourhood U_x that lies entirely within A. Then

$$\{x\} \subset U_x \subset A$$

and we may take the union over all elements of A to conclude that

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x \subset A.$$

This shows that A is the union of open sets, so A itself is open as well.

T5–5. Let (X, T) be a topological space and suppose $A \subset X$ is open. Show that the boundary of A is contained in the complement of A.

Since A is open, its complement $X - A$ is closed, so $\overline{X - A} = X - A$ and

$$\partial A = \overline{A} \cap \overline{X - A} \subset \overline{X - A} = X - A.$$

T5–6. Find two open intervals $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Let $x < y < z$ and consider the intervals $A = (x, y)$ and $B = (y, z)$. In this case,

$$A \cap B = \varnothing, \qquad \overline{A \cap B} = \varnothing, \qquad \overline{A} \cap \overline{B} = [x, y] \cap [y, z] = \{y\}.$$

T6–1. Consider $\mathbb{R}$ with its usual topology and $\mathbb{Z} \subset \mathbb{R}$ with the subspace topology. Show that every subset of $\mathbb{Z}$ is open in $\mathbb{Z}$.

If the set $A \subset \mathbb{Z}$ contains a single integer x, then this set can be written as

$$A = \{x\} = (x - 1, x + 1) \cap \mathbb{Z}.$$

Since the interval $(x - 1, x + 1)$ is open in $\mathbb{R}$, the set A is thus open in $\mathbb{Z}$. If the set $A \subset \mathbb{Z}$ is arbitrary, then one may express it as the union of its elements, namely

$$A = \bigcup_{x \in A} \{x\}.$$

The sets on the right hand side are all open by above, so their union A is open as well.

T6–2. For which topology on X is every function $f\colon X \to Y$ continuous? For which topology on Y is every function $f\colon X \to Y$ continuous?

To say that f is continuous is to say that $f^{-1}(U)$ is open in X for each set U which is open in Y. Suppose that X has the discrete topology. Then every subset of X is open in X, so the inverse image is always open and f is continuous. Similarly, suppose that Y has the indiscrete topology. Then the only subsets of Y which are open are $\varnothing, Y$ and their inverse images are $\varnothing, X$ which are both open in X. Thus, f is continuous in that case as well.

T6–3. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is open in $\mathbb{R}^2$.

Consider the projection on the first variable $p_1\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $p_1(x, y) = x$. This function is continuous by Theorem 2.12 and $(0, \infty)$ is open in $\mathbb{R}$, so its inverse image

$$p_1^{-1}(0, \infty) = \{(x, y) \in \mathbb{R}^2 : p_1(x, y) > 0\}$$

is open in $\mathbb{R}^2$. It remains to note that this inverse image is equal to A, namely

$$p_1^{-1}(0, \infty) = \{(x, y) \in \mathbb{R}^2 : x > 0\} = A.$$

T6–4. Suppose $A \subset X$ and $B \subset Y$. Show that the interior of $A \times B$ in the product space $X \times Y$ is given by $(A \times B)^\circ = A^\circ \times B^\circ$.

If a point (x, y) lies in the interior of $A \times B$, then there is an open set W in $X \times Y$ with

$$(x, y) \in W \subset A \times B.$$

In view of the definition of the product topology, this actually means that

$$(x, y) \in \bigcup_i (U_i \times V_i) \subset A \times B$$

for some sets U_i which are open in X and some sets V_i which are open in Y. Thus,

$$x \in \bigcup_i U_i \subset A \quad \Longrightarrow \quad x \in A^\circ$$

and also $y \in B^\circ$ for similar reasons. Conversely, suppose that $x \in A^\circ$ and $y \in B^\circ$. Then there exists an open set U in X and an open set V in Y such that

$$x \in U \subset A, \qquad y \in V \subset B.$$

This gives $(x, y) \in U \times V \subset A \times B$, so the point (x, y) lies in the interior of $A \times B$.

T6–5. Suppose X is Hausdorff and let $x \in X$ be arbitrary. What is the intersection of all the open sets that contain x?

The intersection must obviously contain x, but it does not contain any other point. In fact, each point $y \neq x$ gives rise to open sets U, V such that

$$x \in U, \qquad y \in V, \qquad U \cap V = \varnothing.$$

This means that x has a neighbourhood U which does not contain the point y.

T6–6. Suppose $f: X \to Y$ is both continuous and injective. Suppose also that Y is Hausdorff. Show that X must be Hausdorff as well.

If x, y are distinct points in X, then $f(x), f(y)$ are distinct points in Y by injectivity. Since Y is Hausdorff, there exist sets U, V which are open in Y such that

$$f(x) \in U, \qquad f(y) \in V, \qquad U \cap V = \varnothing.$$

Consider their inverse images $f^{-1}(U)$ and $f^{-1}(V)$. These are open in X with

$$x \in f^{-1}(U), \qquad y \in f^{-1}(V).$$

To show that they are also disjoint, we note that

$$x \in f^{-1}(U) \cap f^{-1}(V) \quad \Longrightarrow \quad f(x) \in U \cap V = \varnothing.$$

This proves that x, y have disjoint neighbourhoods in X, so X is Hausdorff as well.

T7–1. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.

In terms of polar coordinates, the points on the unit circle are the points that have the form $(\cos\theta, \sin\theta)$. Let us then consider the function

$$f: [0, 2\pi) \to \mathbb{R}^2, \qquad f(\theta) = (\cos\theta, \sin\theta).$$

Each coordinate of f is continuous, so f itself is continuous. Since the interval $[0, 2\pi)$ is connected, the image of f is connected as well, so the unit circle A is connected.

T7–2. Show that there is no continuous bijection $f: [0, 1) \to \mathbb{R}$.

Suppose that f is such a bijection and consider its restriction

$$g: (0, 1) \to \mathbb{R}, \qquad g(x) = f(x).$$

This is continuous by Theorem 2.11 and the interval $(0, 1)$ is connected, so the image of g must be connected. On the other hand, the image of g is given by

$$\mathbb{R} - \{f(0)\} = (-\infty, f(0)) \cup (f(0), \infty).$$

This is a subset of $\mathbb{R}$ which is not an interval, so the image of g is not connected.

T7–3. Suppose $A_1, A_2, \ldots, A_n, B$ are connected and $A_i \cap B$ is nonempty for each i. Show that the union $A_1 \cup A_2 \cup \cdots \cup A_n \cup B$ is connected.

Since A_i and B have a point in common, $C_i = A_i \cup B$ is connected for each i. Note that the sets C_i have a point in common because they all contain B. Thus, the union of these sets is connected as well. It remains to note that the union of the sets C_i is

$$C_1 \cup C_2 \cup \cdots \cup C_n = A_1 \cup A_2 \cup \cdots \cup A_n \cup B.$$

T7–4. Suppose that $f\colon X \to Y$ is continuous and $A \subset Y$ is connected. Does the inverse image $f^{-1}(A)$ have to be connected?

No. Consider the constant function $f\colon X \to \{0\}$ when $X = (0,1) \cup (2,3)$. In this case, the set $A = \{0\}$ is certainly connected, but its inverse image is X which is not connected.

T7–5. Show that there is no continuous bijection $f\colon (0,1) \to A$ when

$$A = \{(x,y) \in \mathbb{R}^2 : xy = 0\}.$$

To say that $xy = 0$ is to say that one of x, y is zero. This means that A is the union of the two axes in the xy-plane, so one can write $A = A_x \cup A_y$ with

$$A_x = \mathbb{R} \times \{0\}, \qquad A_y = \{0\} \times \mathbb{R}.$$

The axes A_x, A_y are connected and they have the origin in common, so their union A is connected. If we remove the origin from A, however, the resulting set is not connected. In fact, the resulting set is the union of the four connected components

$$(-\infty, 0) \times \{0\}, \qquad (0,\infty) \times \{0\}, \qquad \{0\} \times (-\infty, 0), \qquad \{0\} \times (0,\infty).$$

These are all connected because they are products of connected sets.

Suppose now that $f\colon (0,1) \to A$ is a bijection and consider the unique point $x_0 \in (0,1)$ which maps to the origin in A. According to Theorem 2.11, the restriction

$$g\colon (0, x_0) \cup (x_0, 1) \to A$$

is continuous. Now, the image of this function is the union of four connected components. Since the interval $(0, x_0)$ is connected, its image is connected as well, so it must lie entirely within a single component. The same is true for the image of $(x_0, 1)$, so the image of the original function f is a proper subset of A. This contradicts the fact that f is bijective.

T7–6. Suppose A, B are open subsets of X such that $A \cap B, A \cup B$ are both connected. Show that A, B must be connected as well.

Since the roles of A and B are interchangeable, it suffices to show that A is connected. Suppose that $A = U \cup V$ is a partition of A. The set $A \cap B$ is a connected subset of this partition, so it must lie entirely within either U or V. Assume $A \cap B \subset U$ without loss of generality and consider the sets $U \cup B$ and V. These are nonempty and open with

$$(U \cup B) \cup V = (U \cup V) \cup B = A \cup B,$$
$$(U \cup B) \cap V = (U \cap V) \cup (B \cap V) = \varnothing \cup \varnothing = \varnothing.$$

This is actually a contradiction because the set $A \cup B$ is connected by assumption.

T8–1. Show that $A = \{(x, y) \in \mathbb{R}^2 : x^2 + \sin y \leq 1\}$ is not compact.

To say that $A \subset \mathbb{R}^2$ is compact is to say that A is closed and bounded. In this case, A is not bounded because $\sin y \leq 1$ for all $y \in \mathbb{R}$ and thus $(0, y) \in A$ for all $y \in \mathbb{R}$.

T8–2. Show that $B = \{(x, y) \in \mathbb{R}^2 : x^4 - 2x^2 + y^2 \leq 3\}$ is compact.

To say that $B \subset \mathbb{R}^2$ is compact is to say that B is closed and bounded. To prove the first part, consider the function defined by $f(x, y) = x^4 - 2x^2 + y^2$ and note that

$$B = \{(x, y) \in \mathbb{R}^2 : f(x, y) \leq 3\} = f^{-1}(I), \qquad I = (-\infty, 3].$$

Since I is closed in $\mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, the inverse image $B = f^{-1}(I)$ must be closed in $\mathbb{R}^2$. To prove that B is also bounded, we complete the square to get

$$(x^2 - 1)^2 + y^2 = x^4 - 2x^2 + y^2 + 1 \leq 4.$$

This obviously implies that $y^2 \leq 4$, hence $|y| \leq 2$, but it also implies that

$$|x^2 - 1| \leq 2 \quad \Longrightarrow \quad |x^2| \leq |x^2 - 1| + 1 \leq 3 \quad \Longrightarrow \quad |x| \leq \sqrt{3}.$$

T8–3. Show that the union of two compact spaces is compact.

Suppose that A, B are compact subsets of a topological space X. To show that their union $A \cup B$ is compact, consider some open sets U_i that cover $A \cup B$. These form an open cover of $A \subset A \cup B$, so a finite number of them covers A by compactness. Similarly, a finite number of the sets U_i covers B, so it only takes a finite number of them to cover $A \cup B$.

T8–4. Find a topological space (X, T) and a compact subset $A \subset X$ such that A is not closed in X. Can such a space X be Hausdorff?

If the space X is Hausdorff and $A \subset X$ is compact, then A is closed by Theorem 2.19. This means that we need a space X which is not Hausdorff, hence a space X which is not a metric space. The simplest such example is a space $X = \{1, 2\}$ that has two elements with the indiscrete topology $T = \{\varnothing, X\}$. This is not Hausdorff because the points $1, 2$ have no disjoint neighbourhoods. Consider the subset $A = \{1\}$. It is obviously compact, but it is not closed because its complement $\{2\}$ is not open.

T8–5. Suppose $\{x_n\}$ is a sequence in a topological space that converges to the point x. Show that the set $A = \{x, x_1, x_2, x_3, \ldots\}$ is compact.

Suppose the sets U_i form an open cover of A and let U_{i_0} be the set which contains the point x. Since U_{i_0} is open and the sequence $\{x_n\}$ converges to x, there exists an integer N such that $x_n \in U_{i_0}$ for all $n \geq N$. Thus, U_{i_0} contains all the terms except possibly for the terms $x_1, x_2, \ldots, x_{N-1}$. Choose an open set U_{i_k} that contains x_k for each $1 \leq k < N$. Then these $N - 1$ sets together with U_{i_0} form a finite subcover of A.

T8–6. Let C_n be a sequence of nonempty, closed subsets of a compact space X such that $C_n \supset C_{n+1}$ for each n and let A be an open set that contains $\bigcap C_n$. Show that A contains C_k for some k.

Since C_n is closed, its complement $U_n = X - C_n$ is open and we also have

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X - C_n) = X - \bigcap_{n=1}^{\infty} C_n.$$

Thus, the open sets U_n cover the whole space X except for the intersection $\bigcap C_n$. Since the intersection is covered by the set A, we conclude that A together with the sets U_n form an open cover of X. It follows by compactness that finitely many sets cover X, say

$$X = A \cup U_1 \cup U_2 \cup \cdots \cup U_k.$$

Using De Morgan's law once again, one may express the union of the sets U_i as

$$\bigcup_{n=1}^{k} U_n = \bigcup_{n=1}^{k} (X - C_n) = X - \bigcap_{n=1}^{k} C_n = X - C_k.$$

Once we now combine the last two equations, we may finally conclude that

$$C_k \subset X = A \cup (X - C_k) \quad \Longrightarrow \quad C_k \subset A.$$