MA2223 – Tutorial solutions
Part 2. Topological spaces

**T5–1.** Let $T$ be the collection of subsets of $\mathbb{R}$ that consists of $\emptyset, \mathbb{R}$ and every interval of the form $(-\infty, a)$. Show that $T$ is a topology on $\mathbb{R}$.

We check the properties that a topology needs to satisfy. First of all, the sets $\emptyset, \mathbb{R}$ are open by assumption. To show that unions of open sets are open, we note that

$$\bigcup_i (-\infty, a_i) = (-\infty, \sup_i a_i).$$

Given any number of intervals that have the form $(-\infty, a_i)$, their union is then an interval of the same form, so it is open as well. The same is true for finite intersections because

$$\bigcap_{i=1}^n (-\infty, a_i) = (-\infty, \min_{1 \leq i \leq n} a_i).$$

**T5–2.** Find the closure of $(0, 1) \subset \mathbb{R}$ with respect to the discrete topology, the indiscrete topology and the topology of the previous problem.

By definition, the closure of $A$ is the smallest closed set that contains $A$. If we use the discrete topology, then every set is open, so every set is closed. This implies that $\overline{A} = A$. If we use the indiscrete topology, then only $\emptyset, \mathbb{R}$ are open, so only $\emptyset, \mathbb{R}$ are closed and this implies that $\overline{A} = \mathbb{R}$. As for the topology of the previous problem, the nontrivial closed sets have the form $[a, \infty)$ and the smallest one that contains $A = (0, 1)$ is the set $\overline{A} = [0, \infty)$.

**T5–3.** Consider $\mathbb{R}^2$ with its usual topology. Find the closure, the interior and the boundary of the upper half plane $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

The upper half plane $A$ is open, so it is equal to its own interior, namely $A^\circ = A$. The closure must contain the points which are limits of sequences in $A$, so the closure is

$$\overline{A} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}.$$

Finally, the boundary of $A$ is defined as $\partial A = \overline{A} \cap \overline{X-A}$. In this case, we have

$$X - A = \{(x, y) \in \mathbb{R}^2 : y \leq 0\} \implies \overline{X-A} = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$$

$$\implies \partial A = \{(x, y) \in \mathbb{R}^2 : y = 0\}.$$
**T5–4.** Let \((X, T)\) be a topological space and \(A \subset X\). Show that \(A\) is open if and only if each \(x \in A\) has a neighbourhood \(U\) such that \(U \subset A\).

If the set \(A\) is open, then \(A = A^o\) by Theorem 2.4 and one may use Theorem 2.5 to conclude that each \(x \in A\) has a neighbourhood \(U_x\) that lies entirely within \(A\). Conversely, suppose that each \(x \in A\) has a neighbourhood \(U_x\) that lies entirely within \(A\). Then

\[
\{x\} \subset U_x \subset A
\]

and we may take the union over all elements of \(A\) to conclude that

\[
A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x \subset A.
\]

This shows that \(A\) is the union of open sets, so \(A\) itself is open as well.

**T5–5.** Let \((X, T)\) be a topological space and suppose \(A \subset X\) is open. Show that the boundary of \(A\) is contained in the complement of \(A\).

Since \(A\) is open, its complement \(X - A\) is closed, so \(X - A = X - A\) and

\[
\partial A = \overline{A} \cap \overline{X - A} \subset \overline{X - A} = X - A.
\]

**T5–6.** Find two open intervals \(A, B \subset \mathbb{R}\) such that \(\overline{A \cap B} \neq \overline{A} \cap \overline{B}\).

Let \(x < y < z\) and consider the intervals \(A = (x, y)\) and \(B = (y, z)\). In this case,

\[
A \cap B = \emptyset, \quad \overline{A \cap B} = \emptyset, \quad \overline{A} \cap \overline{B} = [x, y] \cap [y, z] = \{y\}.
\]

**T6–1.** Consider \(\mathbb{R}\) with its usual topology and \(\mathbb{Z} \subset \mathbb{R}\) with the subspace topology. Show that every subset of \(\mathbb{Z}\) is open in \(\mathbb{Z}\).

If the set \(A \subset \mathbb{Z}\) contains a single integer \(x\), then this set can be written as

\[
A = \{x\} = (x - 1, x + 1) \cap \mathbb{Z}.
\]

Since the interval \((x - 1, x + 1)\) is open in \(\mathbb{R}\), the set \(A\) is thus open in \(\mathbb{Z}\). If the set \(A \subset \mathbb{Z}\) is arbitrary, then one may express it as the union of its elements, namely

\[
A = \bigcup_{x \in A} \{x\}.
\]

The sets on the right hand side are all open by above, so their union \(A\) is open as well.
T6–2. For which topology on $X$ is every function $f : X \to Y$ continuous? For which topology on $Y$ is every function $f : X \to Y$ continuous?

To say that $f$ is continuous is to say that $f^{-1}(U)$ is open in $X$ for each set $U$ which is open in $Y$. Suppose that $X$ has the discrete topology. Then every subset of $X$ is open in $X$, so the inverse image is always open and $f$ is continuous. Similarly, suppose that $Y$ has the indiscrete topology. Then the only subsets of $Y$ which are open are $\emptyset, Y$ and their inverse images are $\emptyset, X$ which are both open in $X$. Thus, $f$ is continuous in that case as well.

T6–3. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is open in $\mathbb{R}^2$.

Consider the projection on the first variable $p_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $p_1(x, y) = x$. This function is continuous by Theorem 2.12 and $(0, \infty)$ is open in $\mathbb{R}$, so its inverse image

$$p_1^{-1}(0, \infty) = \{(x, y) \in \mathbb{R}^2 : p_1(x, y) > 0\}$$

is open in $\mathbb{R}^2$. It remains to note that this inverse image is equal to $A$, namely

$$p_1^{-1}(0, \infty) = \{(x, y) \in \mathbb{R}^2 : x > 0\} = A.$$

T6–4. Suppose $A \subset X$ and $B \subset Y$. Show that the interior of $A \times B$ in the product space $X \times Y$ is given by $(A \times B)^{\circ} = A^{\circ} \times B^{\circ}$.

If a point $(x, y)$ lies in the interior of $A \times B$, then there is an open set $W$ in $X \times Y$ with

$$(x, y) \in W \subset A \times B.$$

In view of the definition of the product topology, this actually means that

$$(x, y) \in \bigcup_i (U_i \times V_i) \subset A \times B$$

for some sets $U_i$ which are open in $X$ and some sets $V_i$ which are open in $Y$. Thus,

$$x \in \bigcup_i U_i \subset A \quad \Rightarrow \quad x \in A^{\circ}$$

and also $y \in B^{\circ}$ for similar reasons. Conversely, suppose that $x \in A^{\circ}$ and $y \in B^{\circ}$. Then there exists an open set $U$ in $X$ and an open set $V$ in $Y$ such that

$$x \in U \subset A, \quad y \in V \subset B.$$

This gives $(x, y) \in U \times V \subset A \times B$, so the point $(x, y)$ lies in the interior of $A \times B$. 
T6–5. Suppose $X$ is Hausdorff and let $x \in X$ be arbitrary. What is the intersection of all the open sets that contain $x$?

The intersection must obviously contain $x$, but it does not contain any other point. In fact, each point $y \neq x$ gives rise to open sets $U, V$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

This means that $x$ has a neighbourhood $U$ which does not contain the point $y$.

T6–6. Suppose $f : X \to Y$ is both continuous and injective. Suppose also that $Y$ is Hausdorff. Show that $X$ must be Hausdorff as well.

If $x, y$ are distinct points in $X$, then $f(x), f(y)$ are distinct points in $Y$ by injectivity. Since $Y$ is Hausdorff, there exist sets $U, V$ which are open in $Y$ such that

$$f(x) \in U, \quad f(y) \in V, \quad U \cap V = \emptyset.$$

Consider their inverse images $f^{-1}(U)$ and $f^{-1}(V)$. These are open in $X$ with

$$x \in f^{-1}(U), \quad y \in f^{-1}(V).$$

To show that they are also disjoint, we note that

$$x \in f^{-1}(U) \cap f^{-1}(V) \implies f(x) \in U \cap V = \emptyset.$$

This proves that $x, y$ have disjoint neighbourhoods in $X$, so $X$ is Hausdorff as well.

T7–1. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.

In terms of polar coordinates, the points on the unit circle are the points that have the form $(\cos \theta, \sin \theta)$. Let us then consider the function

$$f : [0, 2\pi) \to \mathbb{R}^2, \quad f(\theta) = (\cos \theta, \sin \theta).$$

Each coordinate of $f$ is continuous, so $f$ itself is continuous. Since the interval $[0, 2\pi)$ is connected, the image of $f$ is connected as well, so the unit circle $A$ is connected.

T7–2. Show that there is no continuous bijection $f : [0, 1) \to \mathbb{R}$.

Suppose that $f$ is such a bijection and consider its restriction

$$g : (0, 1) \to \mathbb{R}, \quad g(x) = f(x).$$

This is continuous by Theorem 2.11 and the interval $(0, 1)$ is connected, so the image of $g$ must be connected. On the other hand, the image of $g$ is given by

$$\mathbb{R} - \{f(0)\} = (-\infty, f(0)) \cup (f(0), \infty).$$

This is a subset of $\mathbb{R}$ which is not an interval, so the image of $g$ is not connected.
T7–3. Suppose $A_1, A_2, \ldots, A_n, B$ are connected and $A_i \cap B$ is nonempty for each $i$. Show that the union $A_1 \cup A_2 \cup \cdots \cup A_n \cup B$ is connected.

Since $A_i$ and $B$ have a point in common, $C_i = A_i \cup B$ is connected for each $i$. Note that the sets $C_i$ have a point in common because they all contain $B$. Thus, the union of these sets is connected as well. It remains to note that the union of the sets $C_i$ is

$$C_1 \cup C_2 \cup \cdots \cup C_n = A_1 \cup A_2 \cup \cdots \cup A_n \cup B.$$ 

T7–4. Suppose that $f: X \to Y$ is continuous and $A \subset Y$ is connected. Does the inverse image $f^{-1}(A)$ have to be connected?

No. Consider the constant function $f: X \to \{0\}$ when $X = (0,1) \cup (2,3)$. In this case, the set $A = \{0\}$ is certainly connected, but its inverse image is $X$ which is not connected.

T7–5. Show that there is no continuous bijection $f: (0,1) \to A$ when

$$A = \{(x,y) \in \mathbb{R}^2 : xy = 0\}.$$ 

To say that $xy = 0$ is to say that one of $x, y$ is zero. This means that $A$ is the union of the two axes in the $xy$-plane, so one can write $A = A_x \cup A_y$ with

$$A_x = \mathbb{R} \times \{0\}, \quad A_y = \{0\} \times \mathbb{R}.$$ 

The axes $A_x, A_y$ are connected and they have the origin in common, so their union $A$ is connected. If we remove the origin from $A$, however, the resulting set is not connected. In fact, the resulting set is the union of the four connected components

$$(-\infty,0) \times \{0\}, \quad (0,\infty) \times \{0\}, \quad \{0\} \times (-\infty,0), \quad \{0\} \times (0,\infty).$$

These are all connected because they are products of connected sets.

Suppose now that $f: (0,1) \to A$ is a bijection and consider the unique point $x_0 \in (0,1)$ which maps to the origin in $A$. According to Theorem 2.11, the restriction

$$g: (0,x_0) \cup (x_0,1) \to A$$

is continuous. Now, the image of this function is the union of four connected components. Since the interval $(0,x_0)$ is connected, its image is connected as well, so it must lie entirely within a single component. The same is true for the image of $(x_0,1)$, so the image of the original function $f$ is a proper subset of $A$. This contradicts the fact that $f$ is bijective.
T7–6. Suppose $A, B$ are open subsets of $X$ such that $A \cap B, A \cup B$ are both connected. Show that $A, B$ must be connected as well.

Since the roles of $A$ and $B$ are interchangeable, it suffices to show that $A$ is connected. Suppose that $A = U \cup V$ is a partition of $A$. The set $A \cap B$ is a connected subset of this partition, so it must lie entirely within either $U$ or $V$. Assume $A \cap B \subset U$ without loss of generality and consider the sets $U \cup B$ and $V$. These are nonempty and open with

\[
(U \cup B) \cap V = (U \cap V) \cup (B \cap V) = \emptyset \cup \emptyset = \emptyset.
\]

This is actually a contradiction because the set $A \cup B$ is connected by assumption.

T8–1. Show that $A = \{(x, y) \in \mathbb{R}^2 : x^2 + \sin y \leq 1\}$ is not compact.

To say that $A \subset \mathbb{R}^2$ is compact is to say that $A$ is closed and bounded. In this case, $A$ is not bounded because $\sin y \leq 1$ for all $y \in \mathbb{R}$ and thus $(0, y) \in A$ for all $y \in \mathbb{R}$.

T8–2. Show that $B = \{(x, y) \in \mathbb{R}^2 : x^4 - 2x^2 + y^2 \leq 3\}$ is compact.

To say that $B \subset \mathbb{R}^2$ is compact is to say that $B$ is closed and bounded. To prove the first part, consider the function defined by $f(x, y) = x^4 - 2x^2 + y^2$ and note that

\[
B = \{(x, y) \in \mathbb{R}^2 : f(x, y) \leq 3\} = f^{-1}(I), \quad I = (-\infty, 3].
\]

Since $I$ is closed in $\mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, the inverse image $B = f^{-1}(I)$ must be closed in $\mathbb{R}^2$. To prove that $B$ is also bounded, we complete the square to get

\[
(x^2 - 1)^2 + y^2 = x^4 - 2x^2 + y^2 + 1 \leq 4.
\]

This obviously implies that $y^2 \leq 4$, hence $|y| \leq 2$, but it also implies that

\[
|x^2 - 1| \leq 2 \implies |x^2| \leq |x^2 - 1| + 1 \leq 3 \implies |x| \leq \sqrt{3}.
\]

T8–3. Show that the union of two compact spaces is compact.

Suppose that $A, B$ are compact subsets of a topological space $X$. To show that their union $A \cup B$ is compact, consider some open sets $U_i$ that cover $A \cup B$. These form an open cover of $A \subset A \cup B$, so a finite number of them covers $A$ by compactness. Similarly, a finite number of the sets $U_i$ covers $B$, so it only takes a finite number of them to cover $A \cup B$. 
T8–4. Find a topological space \((X, T)\) and a compact subset \(A \subset X\) such that \(A\) is not closed in \(X\). Can such a space \(X\) be Hausdorff?

If the space \(X\) is Hausdorff and \(A \subset X\) is compact, then \(A\) is closed by Theorem 2.19. This means that we need a space \(X\) which is not Hausdorff, hence a space \(X\) which is not a metric space. The simplest such example is a space \(X = \{1, 2\}\) that has two elements with the indiscrete topology \(T = \{\emptyset, X\}\). This is not Hausdorff because the points 1, 2 have no disjoint neighbourhoods. Consider the subset \(A = \{1\}\). It is obviously compact, but it is not closed because its complement \(\{2\}\) is not open.

T8–5. Suppose \(\{x_n\}\) is a sequence in a topological space that converges to the point \(x\). Show that the set \(A = \{x, x_1, x_2, x_3, \ldots\}\) is compact.

Suppose the sets \(U_i\) form an open cover of \(A\) and let \(U_{i_0}\) be the set which contains the point \(x\). Since \(U_{i_0}\) is open and the sequence \(\{x_n\}\) converges to \(x\), there exists an integer \(N\) such that \(x_n \in U_{i_0}\) for all \(n \geq N\). Thus, \(U_{i_0}\) contains all the terms except possibly for the terms \(x_1, x_2, \ldots, x_{N-1}\). Choose an open set \(U_{i_k}\) that contains \(x_k\) for each \(1 \leq k < N\). Then these \(N-1\) sets together with \(U_{i_0}\) form a finite subcover of \(A\).

T8–6. Let \(C_n\) be a sequence of nonempty, closed subsets of a compact space \(X\) such that \(C_n \supset C_{n+1}\) for each \(n\) and let \(A\) be an open set that contains \(\bigcap C_n\). Show that \(A\) contains \(C_k\) for some \(k\).

Since \(C_n\) is closed, its complement \(U_n = X - C_n\) is open and we also have

\[
\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X - C_n) = X - \bigcap_{n=1}^{\infty} C_n.
\]

Thus, the open sets \(U_n\) cover the whole space \(X\) except for the intersection \(\bigcap C_n\). Since the intersection is covered by the set \(A\), we conclude that \(A\) together with the sets \(U_n\) form an open cover of \(X\). It follows by compactness that finitely many sets cover \(X\), say

\[
X = A \cup U_1 \cup U_2 \cup \cdots \cup U_k.
\]

Using De Morgan’s law once again, one may express the union of the sets \(U_i\) as

\[
\bigcup_{n=1}^{k} U_n = \bigcup_{n=1}^{k} (X - C_n) = X - \bigcap_{n=1}^{k} C_n = X - C_k.
\]

Once we now combine the last two equations, we may finally conclude that

\[
C_k \subset X = A \cup (X - C_k) \quad \implies \quad C_k \subset A.
\]