MA2223 – Tutorial solutions
Part 1. Metric spaces

T1–1. Show that the function $d(x, y) = \sqrt{|x - y|}$ defines a metric on $\mathbb{R}$.

The given function is symmetric and non-negative with $d(x, y) = 0$ if and only if $x = y$. It remains to check that the triangle inequality holds. This is the case because

$$\begin{aligned} d(x, y) = \sqrt{|x - y|} &\leq \sqrt{|x - z| + |z - y|} \\ &\leq \sqrt{|x - z|} + \sqrt{|z - y|} = d(x, z) + d(z, y). \end{aligned}$$

T1–2. Show that $d(x, y) = |x - y|^3$ does not define a metric on $\mathbb{R}$.

The given function is symmetric and non-negative with $d(x, y) = 0$ if and only if $x = y$. However, it does not satisfy the triangle inequality because

$$d(1, 3) = 2^3 > 1^3 + 1^3 = d(1, 2) + d(2, 3).$$

T1–3. Compute the distances $d_1(f, g)$ and $d_\infty(f, g)$ when $f, g \in C[0, 1]$ are the functions defined by $f(x) = x$ and $g(x) = x^3$.

One has $x^3 \leq x$ for all $0 \leq x \leq 1$ and this implies that

$$d_1(f, g) = \int_0^1 |x - x^3| \, dx = \int_0^1 (x - x^3) \, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

To compute the distance $d_\infty(f, g)$, we need to compute the maximum value of

$$h(x) = |f(x) - g(x)| = x - x^3, \qquad 0 \leq x \leq 1.$$

Since $h(0) = h(1) = 0$ and $h'(x) = 1 - 3x^2$, it easily follows that

$$d_\infty(f, g) = \max_{0 \leq x \leq 1} h(x) = h(3^{-1/2}) = 3^{-1/2}(1 - 3^{-1}) = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

T1–4. Compute the distance $d_\infty(x^2, x+2)$ in the space $C[0,2]$.

We need to compute the maximum value of the function

$$h(x) = |x^2 - x - 2|, \qquad 0 \leq x \leq 2.$$

Since $x^2 - x - 2 = (x-2)(x+1) \leq 0$ for each $0 \leq x \leq 2$, we have

$$h(x) = |x^2 - x - 2| = x + 2 - x^2 \quad \Longrightarrow \quad h'(x) = 1 - 2x.$$

This means that $h(x)$ is increasing for $x < 1/2$ and decreasing for $x > 1/2$, so

$$d_\infty(x^2, x+2) = \max_{0 \leq x \leq 2} h(x) = h(1/2) = 9/4.$$

T1–5. Show that $d_1(x^{1/n}, 1) \to 0$ as $n \to \infty$ in the space $C[0,1]$. Does the same statement hold in the case of the d_∞ metric?

Since $x^{1/n} \leq 1$ for each $0 \leq x \leq 1$, the first distance is given by

$$d_1(x^{1/n}, 1) = \int_0^1 (1 - x^{1/n})\, dx = 1 - \frac{1}{1/n + 1}$$

and it converges to $1 - 1 = 0$ as $n \to \infty$. On the other hand, the second distance is

$$d_\infty(x^{1/n}, 1) = \max_{0 \leq x \leq 1} (1 - x^{1/n}) = 1.$$

T1–6. Show that the d_∞ metric in $\mathbb{R}^2$ is the limit of the d_p metric by showing that $\lim_{p\to\infty} \big[|x|^p + |y|^p\big]^{1/p} = \max(|x|, |y|)$ for all $x, y \in \mathbb{R}$.

Suppose first that $|y| < |x|$. Then x is nonzero and $|\frac{y}{x}| < 1$, so

$$\lim_{p\to\infty} \big[|x|^p + |y|^p\big]^{1/p} = \lim_{p\to\infty} |x| \cdot \left[1 + \left|\frac{y}{x}\right|^p\right]^{1/p} = |x|.$$

This settles the case $|y| < |x|$ and the case $|x| < |y|$ is similar. When $|x| = |y|$, we have

$$\lim_{p\to\infty} \big[|x|^p + |y|^p\big]^{1/p} = \lim_{p\to\infty} \big[2|x|^p\big]^{1/p} = \lim_{p\to\infty} 2^{1/p}|x| = |x|.$$

T2–1. Let (X, d) be a metric space. Given a point $x \in X$ and a real number $r > 0$, show that $A = \{y \in X : d(x, y) > r\}$ is open in X.

Let $y \in A$ be given and note that $\varepsilon = d(x, y) - r$ is positive. We claim that $B(y, \varepsilon)$ is contained entirely within A. In fact, one has

$$\begin{aligned} z \in B(y, \varepsilon) \quad &\Longrightarrow \quad d(z, y) < \varepsilon \\ &\Longrightarrow \quad d(z, y) + r < \varepsilon + r = d(x, y) \leq d(x, z) + d(z, y) \\ &\Longrightarrow \quad r < d(x, z) \\ &\Longrightarrow \quad z \in A. \end{aligned}$$

This shows that $B(y, \varepsilon)$ is contained entirely within A and so A is open.

T2–2. Show that $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2y\}$ is open in $\mathbb{R}^2$.

Completing the square, one may express the given set in the form

$$\begin{aligned} A &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2y < 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 1\right\} = B((0, 1), 1). \end{aligned}$$

In particular, A is open in $\mathbb{R}^2$ because every open ball is open in $\mathbb{R}^2$.

T2–3. Show that $A = \{x \in \mathbb{R} : x^3 + 2x^2 - 3x \leq 0\}$ is closed in $\mathbb{R}$.

It suffices to show that the complement of A is open in $\mathbb{R}$. Since

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) = x(x + 3)(x - 1),$$

one has $x^3 + 2x^2 - 3x > 0$ if and only if $x \in (-3, 0) \cup (1, \infty)$. Thus, the complement of A is the union of two open sets, so the complement of A is open and A is closed.

T2–4. Consider the set $X = A \cup B$, where $A = (0, 1)$ and $B = [2, 3)$. Show that A, B are both open in X and thus closed in X as well.

The set A is open because it is the open ball $B(1/2, 1/2)$. Similarly, the set B is open because it is the open ball $B(5/2, 1)$. Since the sets A and B are the complements of one another, the fact that they are both open implies that they are both closed as well.

T2–5. Find a sequence of closed intervals $C_1 \subset C_2 \subset C_3 \subset \cdots$ such that their union is an open interval.

Consider the closed intervals $C_n = [1/n, 3 - 1/n]$, for instance. Since the left endpoint is decreasing to 0 and the right one is increasing to 3, the union of the intervals is $(0, 3)$.

T2–6. Let (X, d) be a metric space whose metric d is discrete. What can you say about a sequence $\{x_n\}$ which is convergent in X?

Suppose the sequence converges to x. Then there exists an integer $N > 0$ such that

$$d(x_n, x) < 1 \quad \text{for all } n \geq N.$$

In particular, $x_n = x$ for all $n \geq N$, so the sequence is eventually constant.

T3–1. Suppose (X, d) is a metric space and $f\colon X \to \mathbb{R}$ is continuous. Show that $A = \{x \in X : |f(x)| < r\}$ is open in X for each $r > 0$.

The given set can be expressed in the form

$$A = \{x \in X : -r < f(x) < r\} = f^{-1}(-r, r).$$

Since $(-r, r)$ is open in $\mathbb{R}$, its inverse image must then be open in X by continuity.

T3–2. Show that every function $f\colon X \to Y$ is continuous when X, Y are metric spaces and the metric on X is discrete.

To show that f is continuous at x, let $\varepsilon > 0$ be given and take $\delta = 1$. Then

$$d_X(x, y) < \delta \quad \Longrightarrow \quad x = y \quad \Longrightarrow \quad d_Y(f(x), f(y)) = d_Y(f(x), f(x)) = 0 < \varepsilon.$$

T3–3. Suppose $f\colon X \to Y$ is a constant function between metric spaces, say $f(x) = y_0$ for all $x \in X$. Show that f is continuous.

To show that f is continuous at x, let $\varepsilon > 0$ be given and $\delta > 0$ be arbitrary. Then

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) = d_Y(y_0, y_0) = 0 < \varepsilon.$$

T3–4. Show that $f(x) = \cos(\sin(2x))$ is Lipschitz continuous on $[0, 1]$.

This follows immediately by Theorem 1.8 because f is differentiable and

$$|f'(x)| = |\sin(\sin(2x))| \cdot |\cos(2x)| \cdot 2 \leq 2.$$

T3–5. Show that $f_n(x) = \frac{x}{x^2+n^2}$ converges uniformly on $[0, \infty)$.

The given sequence converges pointwise to the zero function $f(x) = 0$. To show that the convergence is uniform, one needs to show that

$$\sup_{x\geq 0} |f_n(x) - f(x)| = \sup_{x\geq 0} f_n(x)$$

goes to zero as $n \to \infty$. Consider the function $f_n(x) = \frac{x}{x^2+n^2}$ on $[0, \infty)$. Since

$$f_n'(x) = \frac{x^2 + n^2 - 2x^2}{(x^2 + n^2)^2} = \frac{n^2 - x^2}{(x^2 + n^2)^2},$$

this function is increasing when $x < n$ and decreasing when $x > n$, so its maximum value is

$$\sup_{x\geq 0} |f_n(x) - f(x)| = \sup_{x\geq 0} f_n(x) = f_n(n) = \frac{n}{2n^2} = \frac{1}{2n}.$$

This expression does go to zero as $n \to \infty$, so the convergence is uniform, indeed.

T3–6. Show that $f_n(x) = \frac{x}{x+n}$ does not converge uniformly on $[0, \infty)$.

The given sequence converges pointwise to the zero function $f(x) = 0$. To show that the convergence is not uniform, we consider the expression

$$\sup_{x\geq 0} |f_n(x) - f(x)| = \sup_{x\geq 0} f_n(x) = \sup_{x\geq 0} \frac{x}{x+n}.$$

As one can easily check, $f_n'(x) = n/(x + n)^2$ is positive, so $f_n(x)$ is increasing and

$$\sup_{x\geq 0} |f_n(x) - f(x)| = \sup_{x\geq 0} \frac{x}{x+n} = \lim_{x\to\infty} \frac{x}{x+n} = 1.$$

This expression does not go to zero as $n \to \infty$, so the convergence is not uniform.

T4–1. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is not complete.

It suffices to find a sequence of points in A such that their limit is not in A. There are obviously lots of examples, but the simplest one is probably

$$(x_n, y_n) = (0, 1/n) \in A, \qquad \lim_{n\to\infty} (x_n, y_n) = (0, 0) \notin A.$$

T4–2. Show that (X, d) is complete, if the metric d is discrete.

Let $\{x_n\}$ be a Cauchy sequence in X. Then there exists an integer $N > 0$ such that

$$d(x_m, x_n) < 1 \quad \text{for all } m, n \geq N.$$

Since the metric d is discrete, this actually gives $x_m = x_n$ for all $m, n \geq N$. Thus,

$$x_m = x_N \quad \text{for all } m \geq N$$

and the given Cauchy sequence converges to the point $x_N \in X$.

T4–3. Let (X, d) be a metric space and suppose $A, B \subset X$ are complete. Show that the union $A \cup B$ is complete as well.

Suppose $\{x_n\}$ is a Cauchy sequence of points in $A \cup B$. There must be infinitely many terms that lie in A or else infinitely many terms that lie in B. Consider the former case, as the other case is similar. Since a subsequence lies in A and A is complete, this subsequence must converge to a point $x \in A$. In view of Theorem 1.13, the original sequence must also converge to x, so it converges to a point $x \in A \cup B$.

T4–4. Show that $f(x) = \ln(x + 2)$ has a unique fixed point in $[0, 2]$.

The function f is increasing, so it maps the interval $[0, 2]$ into the interval

$$[f(0), f(2)] = [\ln 2, \ln 4] \subset [\ln 1, \ln e^2] = [0, 2].$$

Since $[0, 2]$ is closed in $\mathbb{R}$, it is complete by Theorem 1.16. To show that f is a contraction, we use the mean value theorem to write

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y|$$

for some point c between x and y. Since $f'(x) = 1/(x + 2)$, this actually gives

$$|f(x) - f(y)| = \frac{1}{c + 2} \cdot |x - y| \leq \frac{1}{2} \cdot |x - y|.$$

In particular, f is a contraction and the result follows by Banach's fixed point theorem.

T4–5. Show that $f(x) = x^3$ is a contraction on $X = (0, 1/2)$, but it has no fixed point in X. Does this contradict Banach's theorem?

The function f is increasing, so it maps the interval $(0, 1/2)$ into the interval

$$(0, 1/8) \subset (0, 1/2).$$

It does not have any fixed points because $x^3 = x$ implies that $x = 0, \pm 1$ and none of these points lies in X. To show that f is a contraction, we use the mean value theorem to write

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y|$$

for some point c between x and y. Since $f'(x) = 3x^2$, we conclude that

$$|f(x) - f(y)| = 3c^2 \cdot |x - y| \leq \frac{3}{4} \cdot |x - y|.$$

This example does not contradict Banach's theorem because $X = (0, 1/2)$ is not complete.

T4–6. Suppose that $f \colon X \to Y$ is a continuous function and $\{x_n\}$ is a Cauchy sequence in X. Does $\{f(x_n)\}$ have to be Cauchy as well?

No. For instance, $f(x) = 1/x$ is continuous on $(0, \infty)$ and $x_n = 1/n$ is Cauchy, but the sequence $f(x_n) = n$ is not Cauchy because $f(x_n) - f(x_{n-1}) = 1$ for all n.