Tutorial 5. Closure, interior and boundary

1. Let T be the collection of subsets of $\mathbb{R}$ that consists of $\varnothing, \mathbb{R}$ and every interval of the form $(-\infty, a)$. Show that T is a topology on $\mathbb{R}$.

2. Find the closure of $(0,1) \subset \mathbb{R}$ with respect to the discrete topology, the indiscrete topology and the topology of the previous problem.

3. Consider $\mathbb{R}^2$ with its usual topology. Find the closure, the interior and the boundary of the upper half plane $A = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

4. Let (X,T) be a topological space and $A \subset X$. Show that A is open if and only if each $x \in A$ has a neighbourhood U such that $U \subset A$.

5. Let (X,T) be a topological space and suppose $A \subset X$ is open. Show that the boundary of A is contained in the complement of A.

6. Find two open intervals $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Tutorial 5. Some hints

1. The union of the intervals $(-\infty, a_i)$ is equal to $(-\infty, \sup a_i)$, while the intersection of finitely many of them is $(-\infty, \min a_i)$.

2. The answers are $(0, 1)$, $\mathbb{R}$ and $[0, \infty)$, respectively.

3. The closure consists of the points with $y \geq 0$ and the boundary of those with $y = 0$. Since the upper half plane is open, one has $A^\circ = A$.

4. If A is open, then $A = A^\circ$ and one may use Theorem 2.5. If the other condition holds, then each $x \in A$ has a neighbourhood $U_x \subset A$, so $A = \bigcup_{x \in A}\{x\} \subset \bigcup_{x \in A} U_x \subset A$ and A is a union of open sets.

5. Since A is open, one has $\partial A = \overline{A} \cap \overline{X - A} \subset \overline{X - A} = X - A$.

6. Consider two intervals of the form $A = (x, y)$ and $B = (y, z)$.