Homework 9. Solutions

1. Consider the space $X = C[0,1]$ and the sequence $f_n(x) = ne^{-n^2 x}$. Show that $f_n \to 0$ in $(X, ||\cdot||_1)$. Is the same true in $(X, ||\cdot||_2)$?

When it comes to the first norm, we have

$$||f_n||_1 = \int_0^1 ne^{-n^2 x}\, dx = \left[-\frac{e^{-n^2 x}}{n}\right]_{x=0}^1 = \frac{1-e^{-n^2}}{n}$$

and this expression approaches zero as $n \to \infty$. On the other hand,

$$||f_n||_2^2 = \int_0^1 n^2 e^{-2n^2 x}\, dx = \left[-\frac{e^{-2n^2 x}}{2}\right]_{x=0}^1 = \frac{1-e^{-2n^2}}{2}$$

and this expression approaches $1/2$ as $n \to \infty$.

2. Find the closure of the unit ball $B = \{\boldsymbol{x} \in X : ||\boldsymbol{x}|| < 1\}$ when X is a normed vector space. Hint: if $||\boldsymbol{y}|| = 1$, then $(1 - \frac{1}{n})\boldsymbol{y} \in B$.

We claim that the closure of the unit ball B is the unit disk

$$D = \{\boldsymbol{x} \in X : ||\boldsymbol{x}|| \leq 1\}.$$

Since the closure contains B, it contains the points with $||\boldsymbol{x}|| < 1$. Consider a point $\boldsymbol{y}$ with $||\boldsymbol{y}|| = 1$ and the corresponding sequence

$$\boldsymbol{x}_n = (1 - 1/n)\,\boldsymbol{y} \quad \Longrightarrow \quad ||\boldsymbol{x}_n|| = 1 - 1/n < 1.$$

Since the sequence $\boldsymbol{x}_n$ lies in B, its limit $\boldsymbol{y}$ is a limit point of B and so $\boldsymbol{y} \in \overline{B}$. This implies that $D \subset \overline{B}$. On the other hand, D is the inverse image of $(-\infty, 1]$ under a continuous function, so D is a closed set that contains B. We must thus have $\overline{B} \subset D$ as well.

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3. Consider the space $X = C[a,b]$ and let $g \in X$ be fixed. Show that the function $T\colon (X, ||\cdot||_\infty) \to (X, ||\cdot||_1)$ is Lipschitz continuous when

$$T(f(x)) = f(x)g(x) \quad \text{for all } f \in X.$$

In view of the definition of T, one has

$$||T(f_1(x)) - T(f_2(x))||_1 = \int_a^b |f_1(x)g(x) - f_2(x)g(x)|\, dx.$$

Since $|g(x)|$ is continuous on the compact interval $[a,b]$, it attains a maximum value, say M. It easily follows that

$$\begin{aligned} ||T(f_1(x)) - T(f_2(x))||_1 &\leq \int_a^b M||f_1 - f_2||_\infty \, dx \\ &= M(b-a) \cdot ||f_1 - f_2||_\infty. \end{aligned}$$

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4. Show that the function $T\colon \ell^2 \to \mathbb{R}$ is Lipschitz continuous when

$$T(x_1, x_2, x_3, \ldots) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

First of all, we note that

$$|T(\boldsymbol{x}) - T(\boldsymbol{y})| = \left| \sum_{n=1}^{\infty} \frac{1}{n}(x_n - y_n) \right| \leq \sum_{n=1}^{\infty} \frac{1}{n} |x_n - y_n|.$$

Using Hölder's inequality with $p = q = 2$, we conclude that

$$|T(\boldsymbol{x}) - T(\boldsymbol{y})| \leq \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \right]^{1/2} \left[\sum_{n=1}^{\infty} |x_n - y_n|^2 \right]^{1/2} = C||\boldsymbol{x} - \boldsymbol{y}||_2.$$