Homework 5. Solutions

1. Find the interior, the closure and the boundary of the following sets. You need not justify your answers.

$$A = \left\{(x,y) \in \mathbb{R}^2 : xy \geq 0\right\}, \qquad B = \left\{(x,y) \in \mathbb{R}^2 : y \neq x^2\right\}.$$

The set A is closed, so it is equal to its own closure, while

$$A^\circ = \left\{(x,y) \in \mathbb{R}^2 : xy > 0\right\},$$
$$\partial A = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\}.$$

The set B is open, so it is equal to its own interior, while

$$\overline{B} = \mathbb{R}^2, \qquad \partial B = \left\{(x,y) \in \mathbb{R}^2 : y = x^2\right\}.$$

Homework 5. Solutions

2. Let (X, T) be a topological space and let $A \subset X$. Show that

$$\partial A = \varnothing \quad \iff \quad A \text{ is both open and closed in } X.$$

If A is both open and closed in X, then the boundary of A is

$$\partial A = \overline{A} \cap \overline{X - A} = A \cap (X - A) = \varnothing.$$

Conversely, suppose that $\partial A = \varnothing$. Then Theorem 2.6 implies that

$$A^\circ = \overline{A}.$$

Since $A^\circ \subset A \subset \overline{A}$ by definition, these sets are all equal, so

$$A^\circ = A = \overline{A} \quad \implies \quad A \text{ is both open and closed in } X.$$

Homework 5. Solutions

3. Consider $\mathbb{R}$ with its usual topology. Find a set $A \subset \mathbb{R}$ such that A and its interior A° do not have the same closure.

If A is any nonempty set whose interior is empty, then

$$A^\circ = \varnothing \quad \Longrightarrow \quad \overline{A^\circ} = \varnothing.$$

On the other hand, $\overline{A}$ cannot be empty since $A \subset \overline{A}$ by definition.

Some typical examples are thus sets $A = \{x\}$ that only contain one element, sets $A = \{x_1, x_2, \ldots, x_n\}$ that contain finitely many elements, or even $A = \mathbb{Z}$ and $A = \mathbb{Q}$. All of these sets have empty interior because none of them contains an open interval.

Homework 5. Solutions

4. Let (X, T) be a topological space and let $A \subset X$. Show that A is closed in X if and only if A contains its boundary.

If the set A is closed, then $\overline{A} = A$ by Theorem 2.3 and

$$\partial A = \overline{A} \cap \overline{X - A} \subset \overline{A} = A.$$

Conversely, suppose that $\partial A \subset A$. Then Theorem 2.6 gives

$$\overline{A} = A^\circ \cup \partial A \subset A^\circ \cup A \subset A.$$

Since $A \subset \overline{A}$ by definition, this gives $\overline{A} = A$ and so A is closed.

Homework 6. Solutions

1. Suppose X, Y are topological spaces, let $A \subset Y$ and let $i \colon A \to Y$ be the inclusion map. Show that a function $f \colon X \to A$ is continuous if and only if the composition $i \circ f \colon X \to Y$ is continuous.

The inclusion map i is continuous by Theorem 2.10. If we assume that f is continuous, then $i \circ f$ is the composition of continuous functions, so it is continuous by Theorem 2.8.

Conversely, suppose $i \circ f$ is continuous and U is open in A. We can then write $U = V \cap A$ for some set V which is open in Y. Since

$$i^{-1}(V) = \{x \in A : i(x) \in V\} = V \cap A = U,$$

one finds that $f^{-1}(U) = f^{-1}(i^{-1}(V)) = (i \circ f)^{-1}(V)$. Since V is open in Y, this set must be open in X and so f is continuous.

Homework 6. Solutions

2. Suppose $A \subset X$ is closed in X and $B \subset Y$ is closed in Y. Show that $A \times B$ is closed in $X \times Y$. Hint: when is (x, y) not in $A \times B$?

We need to show that the complement of $A \times B$ is open in $X \times Y$. Now, $(x, y) \notin A \times B$ if and only if $x \notin A$ or $y \notin B$. This gives

$$X \times Y - A \times B = (X - A) \times Y \cup X \times (Y - B).$$

Since A is closed in X, its complement $X - A$ is open in X and the set $(X - A) \times Y$ is open in the product space $X \times Y$. Using the same argument, one finds that $X \times (Y - B)$ is open as well. Being the union of open sets, the complement of $A \times B$ is thus open.

Homework 6. Solutions

3. Show that A is open in $X \times X$ when X is Hausdorff and

$$A = \{(x, y) \in X \times X : x \neq y\}.$$

Let (x, y) be an arbitrary point of A. Then $x \neq y$ and there exist sets U, V which are open in X with $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Now, the product $U \times V$ is a neighbourhood of (x, y) such that

$$\begin{aligned} (a, b) \in U \times V \quad &\implies \quad a \in U \text{ and } b \in V \\ &\implies \quad a \neq b \\ &\implies \quad (a, b) \in A. \end{aligned}$$

It is thus a neighbourhood of (x, y) which lies within A. This shows that every element of A is in A°, so $A \subset A^\circ \subset A$ and A is open.

Homework 6. Solutions

4. Suppose $f, g\colon X \to Y$ are continuous and Y is Hausdorff. Show that the set $A = \{x \in X : f(x) \neq g(x)\}$ is open in X.

Let $x \in A$ be arbitrary. Then $f(x) \neq g(x)$ and there exist sets U, V which are open in Y such that

$$f(x) \in U, \qquad g(x) \in V, \qquad U \cap V = \varnothing.$$

Consider the set $W = f^{-1}(U) \cap g^{-1}(V)$. This is open in X and

$$\begin{aligned} y \in W \quad &\implies \quad f(y) \in U \text{ and } g(y) \in V \\ &\implies \quad f(y) \neq g(y) \\ &\implies \quad y \in A. \end{aligned}$$

Thus, W is a neighbourhood of x which lies within A. This shows that every element of A is in A°, so $A \subset A^\circ \subset A$ and A is open.

Homework 7. Solutions

1. Suppose $X = A \cup B$ is a partition of a topological space X and define $f\colon X \to \{0,1\}$ by $f(x) = 0$, if $x \in A$, and $f(x) = 1$, if $x \in B$. Show that the function f is continuous.

To say that f is continuous is to say that $f^{-1}(U)$ is open in X for every set U which is open in $Y = \{0,1\}$. The subsets of Y are

$$\varnothing, \qquad \{0\}, \qquad \{1\}, \qquad \{0,1\}$$

and the corresponding inverse images are

$$\varnothing, \qquad A, \qquad B, \qquad X = A \cup B.$$

The sets $\varnothing, X$ are open in X by definition and the sets A, B are open in X by assumption. This implies that $f^{-1}(U)$ is open in X.

2. Show that the hyperbola H has two connected components.

$$H = \{(x, y) \in \mathbb{R}^2 : xy = 1\}.$$

To say that $xy = 1$ is to say that $x \neq 0$ and $y = 1/x$. We now use this fact to express H as the union of the two sets

$$C_+ = \{(x, 1/x) \in \mathbb{R}^2 : x > 0\},$$
$$C_- = \{(x, 1/x) \in \mathbb{R}^2 : x < 0\}.$$

Note that C_+ is the image of the function $f\colon (0, \infty) \to \mathbb{R}^2$ which is defined by $f(x) = (x, 1/x)$. Since f is continuous and $(0, \infty)$ is connected, C_+ is connected and so is C_- for similar reasons. This shows that H is the union of two connected sets. Were H connected itself, its projection onto the first variable would be connected. This is not the case, however, because $p_1(H) = (-\infty, 0) \cup (0, \infty)$.

3. Show that there is no continuous surjection $f\colon H \to A$ when H is the hyperbola of the previous problem and $A = (0,1) \cup (2,3) \cup (4,5)$.

First of all, we express $H = H_1 \cup H_2$ and $A = A_1 \cup A_2 \cup A_3$ as the disjoint union of connected components. If a function $f\colon H \to A$ is continuous, then its restriction $f\colon H_i \to A$ must be continuous for each i, so the image $f(H_i)$ must be connected as well.

Since $f(H_i)$ is a connected subset of $A_1 \cup A_2 \cup A_3$, it lies within either A_1 or $A_2 \cup A_3$. If it actually lies in $A_2 \cup A_3$, then it lies within either A_2 or A_3. This means that each $f(H_i)$ is contained in a single interval A_j. Thus, the image of f is contained in two intervals A_j, so the image is a proper subset of A and f is not surjective.

4. Let (X,T) be a topological space and suppose $A_1, A_2, \dots, A_n$ are connected subsets of X such that $A_k \cap A_{k+1}$ is nonempty for each k. Show that the union of these sets is connected. Hint: Use induction.

When $n = 1$, the union is equal to A_1 and this set is connected by assumption. Suppose that the result holds for n sets and consider the union of $n+1$ sets. This union has the form

$$U = A_1 \cup \cdots \cup A_n \cup A_{n+1} = B \cup A_{n+1},$$

where B is connected by the induction hypothesis. Since A_{n+1} has a point in common with A_n, it has a point in common with B. In particular, the union $B \cup A_{n+1}$ is connected and the result follows.

Homework 8. Solutions

1. Show that $A = \{(x,y) \in \mathbb{R}^2 : x^4 + (y-1)^2 \leq 1\}$ is compact.

First of all, the set A is bounded because its points satisfy

$$\begin{aligned} x^4 \leq x^4 + (y-1)^2 \leq 1 \quad &\Longrightarrow \quad |x| \leq 1, \\ (y-1)^2 \leq x^4 + (y-1)^2 \leq 1 \quad &\Longrightarrow \quad |y| \leq |y-1| + 1 \leq 2. \end{aligned}$$

To show that A is also closed in $\mathbb{R}^2$, we consider the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = x^4 + (y-1)^2.$$

Since f is continuous and $(-\infty, 1]$ is closed in $\mathbb{R}$, its inverse image is closed in $\mathbb{R}^2$. This means that A is closed in $\mathbb{R}^2$. Since A is both bounded and closed in $\mathbb{R}^2$, we conclude that A is compact.

Homework 8. Solutions

2. Let (X, T) be a topological space whose topology T is discrete. Show that a subset $A \subset X$ is compact if and only if it is finite.

First, suppose that A is finite, say $A = \{x_1, x_2, \ldots, x_n\}$. If some open sets U_i form an open cover of A, then each x_k must belong to one of the sets, say U_{i_k}. This implies that $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ form a finite subcover of A, so the set A is compact.

Conversely, suppose that A is compact and recall that every subset of X is open in the discrete topology. We consider the sets $\{x\}$ for each $x \in A$. These form an open cover of A, so finitely many of them cover A by compactness. It easily follows that

$$A \subset \bigcup_{i=1}^{n} \{x_i\} \subset A \quad \Longrightarrow \quad A = \{x_1, x_2, \ldots, x_n\}.$$

3. Let C_n be a sequence of nonempty, closed subsets of a compact space X such that $C_n \supset C_{n+1}$ for each n. Show that the intersection of these sets is nonempty. Hint: One has $\bigcup(X - C_i) = X - \bigcap C_i$.

Suppose the intersection is empty. Then we actually have

$$X = X - \bigcap_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} (X - C_i),$$

so the sets $X - C_i$ form an open cover of X. Since X is compact, it is covered by finitely many sets, say the first k. This gives

$$X = \bigcup_{i=1}^{k} (X - C_i) = X - \bigcap_{i=1}^{k} C_i = X - C_k,$$

so the set C_k must be empty, contrary to assumption.

4. Suppose $X \subset \mathbb{R}$ is compact and $f\colon X \to X$ is continuous with

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x \neq y.$$

Show that f has a fixed point. Hint: suppose that $g(x) = |f(x) - x|$ attains a positive minimum at the point x_0 and consider $g(f(x_0))$.

Consider the function $g\colon X \to \mathbb{R}$ defined by $g(x) = |f(x) - x|$. It is continuous on a compact set, so it attains a minimum value at some point x_0. If the minimum value is zero, then $f(x_0) = x_0$ and x_0 is a fixed point. Otherwise, $f(x_0) \neq x_0$ and we get

$$g(f(x_0)) = |f(f(x_0)) - f(x_0)| < |f(x_0) - x_0| = g(x_0),$$

contrary to the fact that $g(x_0)$ is the minimum value attained by g.