1. Find the interior, the closure and the boundary of the following sets. You need not justify your answers.

\[ A = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y \neq x^2\}. \]

The set \( A \) is closed, so it is equal to its own closure, while

\[ A^\circ = \{(x, y) \in \mathbb{R}^2 : xy > 0\}, \quad \partial A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}. \]

The set \( B \) is open, so it is equal to its own interior, while

\[ \overline{B} = \mathbb{R}^2, \quad \partial B = \{(x, y) \in \mathbb{R}^2 : y = x^2\}. \]
2. Let \((X, T)\) be a topological space and let \(A \subset X\). Show that

\[
\partial A = \emptyset \iff A \text{ is both open and closed in } X.
\]

If \(A\) is both open and closed in \(X\), then the boundary of \(A\) is

\[
\partial A = \overline{A} \cap (X - A) = A \cap (X - A) = \emptyset.
\]

Conversely, suppose that \(\partial A = \emptyset\). Then Theorem 2.6 implies that

\[
A^\circ = \overline{A}.
\]

Since \(A^\circ \subset A \subset \overline{A}\) by definition, these sets are all equal, so

\[
A^\circ = A = \overline{A} \implies A \text{ is both open and closed in } X.
\]
3. Consider \( \mathbb{R} \) with its usual topology. Find a set \( A \subset \mathbb{R} \) such that \( A \) and its interior \( A^\circ \) do not have the same closure.

If \( A \) is any nonempty set whose interior is empty, then

\[
A^\circ = \emptyset \quad \implies \quad \overline{A^\circ} = \emptyset.
\]

On the other hand, \( \overline{A} \) cannot be empty since \( A \subset \overline{A} \) by definition.

Some typical examples are thus sets \( A = \{x\} \) that only contain one element, sets \( A = \{x_1, x_2, \ldots, x_n\} \) that contain finitely many elements, or even \( A = \mathbb{Z} \) and \( A = \mathbb{Q} \). All of these sets have empty interior because none of them contains an open interval.
4. Let \((X, T)\) be a topological space and let \(A \subset X\). Show that \(A\) is closed in \(X\) if and only if \(A\) contains its boundary.

If the set \(A\) is closed, then \(\overline{A} = A\) by Theorem 2.3 and

\[
\partial A = \overline{A} \cap X - A \subset \overline{A} = A.
\]

Conversely, suppose that \(\partial A \subset A\). Then Theorem 2.6 gives

\[
\overline{A} = A^\circ \cup \partial A \subset A^\circ \cup A \subset A.
\]

Since \(A \subset \overline{A}\) by definition, this gives \(\overline{A} = A\) and so \(A\) is closed.
1. Suppose $X, Y$ are topological spaces, let $A \subset Y$ and let $i: A \to Y$ be the inclusion map. Show that a function $f: X \to A$ is continuous if and only if the composition $i \circ f: X \to Y$ is continuous.

The inclusion map $i$ is continuous by Theorem 2.10. If we assume that $f$ is continuous, then $i \circ f$ is the composition of continuous functions, so it is continuous by Theorem 2.8.

Conversely, suppose $i \circ f$ is continuous and $U$ is open in $A$. We can then write $U = V \cap A$ for some set $V$ which is open in $Y$. Since

$$i^{-1}(V) = \{x \in A : i(x) \in V\} = V \cap A = U,$$

one finds that $f^{-1}(U) = f^{-1}(i^{-1}(V)) = (i \circ f)^{-1}(V)$. Since $V$ is open in $Y$, this set must be open in $X$ and so $f$ is continuous.
2. Suppose $A \subset X$ is closed in $X$ and $B \subset Y$ is closed in $Y$. Show that $A \times B$ is closed in $X \times Y$. Hint: when is $(x, y)$ not in $A \times B$?

We need to show that the complement of $A \times B$ is open in $X \times Y$. Now, $(x, y) \notin A \times B$ if and only if $x \notin A$ or $y \notin B$. This gives

$$X \times Y - A \times B = (X - A) \times Y \cup X \times (Y - B).$$

Since $A$ is closed in $X$, its complement $X - A$ is open in $X$ and the set $(X - A) \times Y$ is open in the product space $X \times Y$. Using the same argument, one finds that $X \times (Y - B)$ is open as well. Being the union of open sets, the complement of $A \times B$ is thus open.
3. Show that $A$ is open in $X \times X$ when $X$ is Hausdorff and

$$A = \{(x, y) \in X \times X : x \neq y\}.$$

Let $(x, y)$ be an arbitrary point of $A$. Then $x \neq y$ and there exist sets $U, V$ which are open in $X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Now, the product $U \times V$ is a neighbourhood of $(x, y)$ such that

$$(a, b) \in U \times V \implies a \in U \text{ and } b \in V$$

$$\implies a \neq b$$

$$\implies (a, b) \in A.$$

It is thus a neighbourhood of $(x, y)$ which lies within $A$. This shows that every element of $A$ is in $A^\circ$, so $A \subset A^\circ \subset A$ and $A$ is open.
4. Suppose $f, g : X \rightarrow Y$ are continuous and $Y$ is Hausdorff. Show that the set $A = \{x \in X : f(x) \neq g(x)\}$ is open in $X$.

Let $x \in A$ be arbitrary. Then $f(x) \neq g(x)$ and there exist sets $U, V$ which are open in $Y$ such that

$$f(x) \in U, \quad g(x) \in V, \quad U \cap V = \emptyset.$$ 

Consider the set $W = f^{-1}(U) \cap g^{-1}(V)$. This is open in $X$ and

$$y \in W \implies f(y) \in U \text{ and } g(y) \in V$$
$$\implies f(y) \neq g(y)$$
$$\implies y \in A.$$

Thus, $W$ is a neighbourhood of $x$ which lies within $A$. This shows that every element of $A$ is in $A^\circ$, so $A \subset A^\circ \subset A$ and $A$ is open.
1. Suppose \( X = A \cup B \) is a partition of a topological space \( X \) and define \( f : X \to \{0, 1\} \) by \( f(x) = 0 \), if \( x \in A \), and \( f(x) = 1 \), if \( x \in B \). Show that the function \( f \) is continuous.

To say that \( f \) is continuous is to say that \( f^{-1}(U) \) is open in \( X \) for every set \( U \) which is open in \( Y = \{0, 1\} \). The subsets of \( Y \) are

\[
\emptyset, \quad \{0\}, \quad \{1\}, \quad \{0, 1\}
\]

and the corresponding inverse images are

\[
\emptyset, \quad A, \quad B, \quad X = A \cup B.
\]

The sets \( \emptyset, X \) are open in \( X \) by definition and the sets \( A, B \) are open in \( X \) by assumption. This implies that \( f^{-1}(U) \) is open in \( X \).
2. Show that the hyperbola $H$ has two connected components.

$$H = \{(x, y) \in \mathbb{R}^2 : xy = 1\}.$$ 

To say that $xy = 1$ is to say that $x \neq 0$ and $y = 1/x$. We now use this fact to express $H$ as the union of the two sets

$$C_+ = \{(x, 1/x) \in \mathbb{R}^2 : x > 0\},$$
$$C_- = \{(x, 1/x) \in \mathbb{R}^2 : x < 0\}.$$ 

Note that $C_+$ is the image of the function $f : (0, \infty) \to \mathbb{R}^2$ which is defined by $f(x) = (x, 1/x)$. Since $f$ is continuous and $(0, \infty)$ is connected, $C_+$ is connected and so is $C_-$ for similar reasons. This shows that $H$ is the union of two connected sets. Were $H$ connected itself, its projection onto the first variable would be connected. This is not the case, however, because $p_1(H) = (-\infty, 0) \cup (0, \infty)$. 
3. Show that there is no continuous surjection $f : H \to A$ when $H$ is the hyperbola of the previous problem and $A = (0, 1) \cup (2, 3) \cup (4, 5)$.

First of all, we express $H = H_1 \cup H_2$ and $A = A_1 \cup A_2 \cup A_3$ as the disjoint union of connected components. If a function $f : H \to A$ is continuous, then its restriction $f : H_i \to A$ must be continuous for each $i$, so the image $f(H_i)$ must be connected as well.

Since $f(H_i)$ is a connected subset of $A_1 \cup A_2 \cup A_3$, it lies within either $A_1$ or $A_2 \cup A_3$. If it actually lies in $A_2 \cup A_3$, then it lies within either $A_2$ or $A_3$. This means that each $f(H_i)$ is contained in a single interval $A_j$. Thus, the image of $f$ is contained in two intervals $A_j$, so the image is a proper subset of $A$ and $f$ is not surjective.
4. Let \((X, T)\) be a topological space and suppose \(A_1, A_2, \ldots, A_n\) are connected subsets of \(X\) such that \(A_k \cap A_{k+1}\) is nonempty for each \(k\). Show that the union of these sets is connected. Hint: Use induction.

When \(n = 1\), the union is equal to \(A_1\) and this set is connected by assumption. Suppose that the result holds for \(n\) sets and consider the union of \(n + 1\) sets. This union has the form

\[
U = A_1 \cup \cdots \cup A_n \cup A_{n+1} = B \cup A_{n+1},
\]

where \(B\) is connected by the induction hypothesis. Since \(A_{n+1}\) has a point in common with \(A_n\), it has a point in common with \(B\). In particular, the union \(B \cup A_{n+1}\) is connected and the result follows.
1. Show that $A = \{(x, y) \in \mathbb{R}^2 : x^4 + (y - 1)^2 \leq 1\}$ is compact.

First of all, the set $A$ is bounded because its points satisfy

\[
x^4 \leq x^4 + (y - 1)^2 \leq 1 \implies |x| \leq 1, \\
(y - 1)^2 \leq x^4 + (y - 1)^2 \leq 1 \implies |y| \leq |y - 1| + 1 \leq 2.
\]

To show that $A$ is also closed in $\mathbb{R}^2$, we consider the function

\[
f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^4 + (y - 1)^2.
\]

Since $f$ is continuous and $(-\infty, 1]$ is closed in $\mathbb{R}$, its inverse image is closed in $\mathbb{R}^2$. This means that $A$ is closed in $\mathbb{R}^2$. Since $A$ is both bounded and closed in $\mathbb{R}^2$, we conclude that $A$ is compact.
2. Let \((X, T)\) be a topological space whose topology \(T\) is discrete. Show that a subset \(A \subset X\) is compact if and only if it is finite.

First, suppose that \(A\) is finite, say \(A = \{x_1, x_2, \ldots, x_n\}\). If some open sets \(U_i\) form an open cover of \(A\), then each \(x_k\) must belong to one of the sets, say \(U_{i_k}\). This implies that \(U_{i_1}, U_{i_2}, \ldots, U_{i_n}\) form a finite subcover of \(A\), so the set \(A\) is compact.

Conversely, suppose that \(A\) is compact and recall that every subset of \(X\) is open in the discrete topology. We consider the sets \(\{x\}\) for each \(x \in A\). These form an open cover of \(A\), so finitely many of them cover \(A\) by compactness. It easily follows that

\[
A \subset \bigcup_{i=1}^{n} \{x_i\} \subset A \implies A = \{x_1, x_2, \ldots, x_n\}.
\]
3. Let \( C_n \) be a sequence of nonempty, closed subsets of a compact space \( X \) such that \( C_n \supset C_{n+1} \) for each \( n \). Show that the intersection of these sets is nonempty. Hint: One has \( \bigcup (X - C_i) = X - \bigcap C_i \).

Suppose the intersection is empty. Then we actually have

\[
X = X - \bigcap_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} (X - C_i),
\]

so the sets \( X - C_i \) form an open cover of \( X \). Since \( X \) is compact, it is covered by finitely many sets, say the first \( k \). This gives

\[
X = \bigcup_{i=1}^{k} (X - C_i) = X - \bigcap_{i=1}^{k} C_i = X - C_k,
\]

so the set \( C_k \) must be empty, contrary to assumption.
4. Suppose $X \subset \mathbb{R}$ is compact and $f: X \rightarrow X$ is continuous with

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x \neq y.$$ 

Show that $f$ has a fixed point. Hint: suppose that $g(x) = |f(x) - x|$ attains a positive minimum at the point $x_0$ and consider $g(f(x_0))$.

Consider the function $g: X \rightarrow \mathbb{R}$ defined by $g(x) = |f(x) - x|$. It is continuous on a compact set, so it attains a minimum value at some point $x_0$. If the minimum value is zero, then $f(x_0) = x_0$ and $x_0$ is a fixed point. Otherwise, $f(x_0) \neq x_0$ and we get

$$g(f(x_0)) = |f(f(x_0)) - f(x_0)| < |f(x_0) - x_0| = g(x_0),$$

contrary to the fact that $g(x_0)$ is the minimum value attained by $g$. 