Homework 1. Solutions

1. Show that the discrete metric satisfies the properties of a metric.

The discrete metric is defined by the formula

$$d(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{array} \right\}.$$

It is clearly symmetric and non-negative with $d(x,y) = 0$ if and only if $x = y$. It remains to establish the triangle inequality

$$d(x,y) \leq d(x,z) + d(z,y).$$

If $x = y$, then the left hand side is zero and the inequality certainly holds. If $x \neq y$, then the left hand side is equal to 1. Since $x \neq y$, we must have either $z \neq x$ or else $z \neq y$. Thus, the right hand side is at least 1 and the triangle inequality holds in any case.

2. Compute the distances $d_2(f,g)$ and $d_\infty(f,g)$ when $f,g \in C[0,1]$ are the functions defined by $f(x) = x$ and $g(x) = x^4$.

Using the definition of the d_2 metric, one finds that

$$\begin{aligned} d_2(f,g)^2 &= \int_0^1 \left(x - x^4\right)^2 \, dx \\ &= \int_0^1 (x^2 - 2x^5 + x^8) \, dx = \frac{1}{3} - \frac{2}{6} + \frac{1}{9} = \frac{1}{9} \end{aligned}$$

and so $d_2(f,g) = \frac{1}{3}$. The distance $d_\infty(f,g)$ is the maximum of

$$h(x) = |x - x^4| = x - x^4, \qquad 0 \le x \le 1.$$

Since $h(0) = h(1) = 0$ and $h'(x) = 1 - 4x^3$, it easily follows that

$$d_\infty(f,g) = h(4^{-1/3}) = 4^{-1/3}(1 - 4^{-1}) = 3 \cdot 4^{-4/3}.$$

Homework 1. Solutions

3. Show that the following functions do not define metrics on $\mathbb{R}$.

$$d(x,y) = |x-2y|, \qquad d(x,y) = (x-y)^2, \qquad d(x,y) = |xy|.$$

The first function does not satisfy any of the desired properties. For instance, $d(1,1) = 1 \neq 0$, while $d(2,1) = 0$ and $d(2,1) \neq d(1,2)$.

The second function does not satisfy the triangle inequality, as

$$d(1,2) + d(2,3) = 1 + 1 < 4 = d(1,3).$$

Finally, the third function is symmetric, but it does not satisfy the other two properties. For instance, $d(0,1) = 0$ and $d(1,1) = 1 \neq 0$.

4. Consider the space $C[0,1]$ with the d_1 metric. For which values of the integer $n \geq 1$ does $f(x) = x^n$ lie in the open ball $B(x, 2/5)$?

To say that x^n lies in the open ball $B(x, 2/5)$ is to say that

$$d_1(x^n, x) < 2/5.$$

Since $x^n \leq x$ for all $x \in [0,1]$, the left hand side is equal to

$$d_1(x^n, x) = \int_0^1 (x - x^n)\, dx = \frac{1}{2} - \frac{1}{n+1}.$$

In particular, the desired condition holds if and only if

$$\frac{1}{2} - \frac{1}{n+1} < \frac{2}{5} \iff \frac{1}{n+1} > \frac{1}{10} \iff n < 9.$$

Homework 2. Solutions

1. Consider the upper half plane $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Use the definition of an open set to show that A is open in $\mathbb{R}^2$.

Let $(x, y) \in A$ be given. Then $y > 0$ and we claim that the open ball $B((x, y), y)$ is contained entirely within A. In fact, one has

$$\begin{aligned}
(a, b) \in B((x, y), y) \quad &\Longrightarrow \quad (x-a)^2 + (y-b)^2 < y^2 \\
&\Longrightarrow \quad (y-b)^2 < y^2 \\
&\Longrightarrow \quad b(b-2y) < 0 \\
&\Longrightarrow \quad 0 < b < 2y \\
&\Longrightarrow \quad (a, b) \in A.
\end{aligned}$$

This shows that $B((x, y), y) \subset A$, so the set A is open in $\mathbb{R}^2$.

Homework 2. Solutions

2. Show that the following set is open in $\mathbb{R}^2$.

$$B = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x \text{ and } y > 0\right\}.$$

The given set is the intersection $B = B_1 \cap B_2$, where

$$B_1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x\right\} \text{ and}$$
$$B_2 = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}.$$

Note that B_2 is the upper half plane and this is open in $\mathbb{R}^2$ by the previous problem. The set B_1 can be expressed in the form

$$B_1 = \left\{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4\right\} = B((2,0),2),$$

so it is an open ball in $\mathbb{R}^2$ and thus open. Being the intersection of two open sets, the given set B is then open as well.

Homework 2. Solutions

3. Show that the following sets are open in $\mathbb{R}$.

$$A = \left\{x \in \mathbb{R} : x^3 > x\right\}, \qquad B = \left\{0 < x < 1 : \frac{1}{x} \notin \mathbb{Z}\right\}.$$

When it comes to the first set, one has

$$x^3 > x \iff x(x^2 - 1) > 0 \iff x(x+1)(x-1) > 0.$$

This implies that $A = (-1, 0) \cup (1, \infty)$ and so A is open in $\mathbb{R}$. The second set is the interval $(0, 1)$ with the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ removed. It is open because it is the union of open intervals, namely

$$B = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \cdots = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right).$$

Homework 2. Solutions

4. Is the set $\mathbb{Q}$ of all rational numbers closed in $\mathbb{R}$? Why or why not?

We use the first part of Theorem 1.4. Were $\mathbb{Q}$ closed in $\mathbb{R}$, every convergent sequence of rational numbers would have to converge to a rational number. However, this is not really the case. As a simple example, consider a rational approximation of $\sqrt{2}$, say

$$\begin{aligned} x_1 &= 1.4, \\ x_2 &= 1.41, \\ x_3 &= 1.414, \\ x_4 &= 1.4142 \end{aligned}$$

and so on. This is a convergent sequence of rational numbers, but its limit $\sqrt{2}$ is not a rational number. Thus, $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Homework 3. Solutions

1. Suppose (X, d) is a metric space and $f\colon X \to \mathbb{R}$ is continuous. Show that the set $A = \{x \in X : f(x) = 0\}$ is closed in X.

We show that the complement of A is open in X. Noting that

$$\begin{aligned} X - A &= \{x \in X : f(x) \neq 0\} \\ &= \{x \in X : f(x) < 0\} \cup \{x \in X : f(x) > 0\}, \end{aligned}$$

we see that $X - A$ can be expressed as the union

$$X - A = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty).$$

Since $(-\infty, 0)$ is open in $\mathbb{R}$, its inverse image must be open in X by continuity. The same is true for the inverse image of $(0, \infty)$. This means that $X - A$ is the union of two open sets and thus open.

2. Show that $f(x) = \sqrt{x^2+1}$ is Lipschitz continuous on $[0,1]$.

According to Theorem 1.8, it suffices to show that

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

is bounded on $[0,1]$. In particular, it suffices to note that

$$|f'(x)| = \sqrt{\frac{x^2}{x^2+1}} < 1.$$

One could also try to find the maximum value of $|f'(x)|$, but this is not really necessary. In fact, one could simply say that $|f'(x)|$ is continuous, so it does attain a maximum value on $[0,1]$.

3. Let (X, d) be a metric space and fix some $y \in X$. Show that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x, y)$ is Lipschitz continuous.

Letting $x, z \in X$ be arbitrary, we use the triangle inequality to get

$$f(x) = d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + f(z)$$
$$f(z) = d(z, y) \leq d(z, x) + d(x, y) = d(x, z) + f(x).$$

Once we now combine these equations, we may conclude that

$$|f(x) - f(z)| \leq d(x, z).$$

This shows that the function $f\colon X \to \mathbb{R}$ is Lipschitz continuous.

4. Consider the sequence of functions defined by $f_n(x) = xe^{-nx}$ for each integer $n \geq 1$. Show that f_n converges uniformly on $[0, 1]$.

When $0 < x \leq 1$, one has $0 < e^{-x} < 1$, so $e^{-nx} \to 0$ as $n \to \infty$. Thus, $f_n(x)$ converges pointwise to the zero function. Since

$$f_n'(x) = e^{-nx} - nxe^{-nx} = (1 - nx)e^{-nx},$$

the function f_n is increasing for $x < \frac{1}{n}$ and decreasing for $x > \frac{1}{n}$, so

$$\sup_{0 \leq x \leq 1} |f_n(x)| = \sup_{0 \leq x \leq 1} f_n(x) = f_n(1/n) = \frac{1}{ne}.$$

In particular, f_n converges uniformly on $[0, 1]$ because

$$\lim_{n \to \infty} \sup_{0 \leq x \leq 1} |f_n(x)| = \lim_{n \to \infty} \frac{1}{ne} = 0.$$

Homework 4. Solutions

1. Which of the following sets are complete? Explain.

$$A = \mathbb{Z}, \qquad B = (0,2), \qquad C = \{x \in \mathbb{R} : \sin x \leq 0\}.$$

According to Theorem 1.16, a subset of $\mathbb{R}$ is complete if and only if it is closed. In our case, $\mathbb{Z}$ is complete because its complement is

$$\mathbb{R} - \mathbb{Z} = \bigcup_{x \in \mathbb{Z}} (x, x+1)$$

and this is open in $\mathbb{R}$. The set B is not complete because $x_n = 1/n$ is a sequence of points in B whose limit is not in B. Finally,

$$\mathbb{R} - C = \{x \in \mathbb{R} : f(x) > 0\} = f^{-1}(0, \infty)$$

with $f(x) = \sin x$ continuous, so $\mathbb{R} - C$ is open and C is complete.

2. Use the definition of a Cauchy sequence to show that $\{(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}^2$, if $\{x_n\}, \{y_n\}$ are Cauchy sequences in $\mathbb{R}$.

Let $\varepsilon > 0$ be given. Then there exist integers $N_1, N_2 > 0$ such that

$$|x_m - x_n| < \varepsilon/\sqrt{2} \quad \text{for all } m, n \geq N_1;$$
$$|y_m - y_n| < \varepsilon/\sqrt{2} \quad \text{for all } m, n \geq N_2.$$

Setting $N = \max\{N_1, N_2\}$ for convenience, we conclude that

$$(x_m - x_n)^2 + (y_m - y_n)^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$$

for all $m, n \geq N$. Thus, $\{(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}^2$.

3. Suppose $f\colon [a,b] \to [a,b]$ is a differentiable function such that

$$L = \sup_{a \leq x \leq b} |f'(x)|$$

satisfies $L < 1$. Show that f has a unique fixed point in $[a,b]$.

Let $x, y \in [a,b]$. Using the mean value theorem, one finds that

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq L \cdot |x - y|$$

for some point c between x and y. Since $L < 1$ by assumption, this shows that f is a contraction on $[a,b]$. On the other hand, $[a,b]$ is a closed subset of $\mathbb{R}$ and thus complete. It follows by Banach's fixed point theorem that f has a unique fixed point in $[a,b]$.

Homework 4. Solutions

4. Show that there is a unique $x \in [1,2]$ such that $x^4 - x - 2 = 0$.

The function $f(x) = (x+2)^{1/4}$ is increasing, so it maps $[1,2]$ into

$$[f(1), f(2)] = [3^{1/4}, 4^{1/4}] \subset [1^{1/4}, 16^{1/4}] = [1,2].$$

To see that the previous problem is applicable, we note that

$$L = \sup_{1 \le x \le 2} |f'(x)| = \sup_{1 \le x \le 2} \frac{1}{4} \cdot (x+2)^{-3/4} = 4^{-1} \cdot 3^{-3/4} < 1.$$

Thus, there exists a unique $x \in [1,2]$ such that $(x+2)^{1/4} = x$ and this is the only point in $[1,2]$ such that $x^4 = x + 2$.