Chapter 3. Normed vector spaces

Proofs covered in class

P. Karageorgis

pete@maths.tcd.ie

Theorem 3.1 – Product norm

Suppose X, Y are normed vector spaces. Then one may define a norm on the product $X \times Y$ by letting $||(\boldsymbol{x}, \boldsymbol{y})|| = ||\boldsymbol{x}|| + ||\boldsymbol{y}||$.

Proof. To see that the given formula defines a norm, we note that

$$||\boldsymbol{x}|| + ||\boldsymbol{y}|| = 0 \quad \iff \quad ||\boldsymbol{x}|| = ||\boldsymbol{y}|| = 0.$$

This implies that $||(\boldsymbol{x}, \boldsymbol{y})|| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y} = 0$, while

$$||(\lambda \boldsymbol{x}, \lambda \boldsymbol{y})|| = ||\lambda \boldsymbol{x}|| + ||\lambda \boldsymbol{y}|| = |\lambda| \cdot ||(\boldsymbol{x}, \boldsymbol{y})||.$$

Adding the triangle inequalities in X and Y, one also finds that

$$||(\boldsymbol{x}_1, \boldsymbol{y}_1) + (\boldsymbol{x}_2, \boldsymbol{y}_2)|| \leq ||(\boldsymbol{x}_1, \boldsymbol{y}_1)|| + ||(\boldsymbol{x}_2, \boldsymbol{y}_2)||.$$

In particular, the triangle inequality holds in $X \times Y$ as well. ■

Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space X.

1. The norm $f(\boldsymbol{x}) = ||\boldsymbol{x}||$, where $\boldsymbol{x} \in X$.

Proof. Using the triangle inequality, one finds that

$$||\boldsymbol{x}|| = ||\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y}|| \leq ||\boldsymbol{x} - \boldsymbol{y}|| + ||\boldsymbol{y}||,$$
$$||\boldsymbol{y}|| = ||\boldsymbol{y} - \boldsymbol{x} + \boldsymbol{x}|| \leq ||\boldsymbol{y} - \boldsymbol{x}|| + ||\boldsymbol{x}||$$

for all $\boldsymbol{x}, \boldsymbol{y} \in X$. Rearranging terms now gives

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq ||\boldsymbol{x} - \boldsymbol{y}||$$

and this makes f Lipschitz continuous, hence also continuous. ■

Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space X.

2. The vector addition $g(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in X$.

Proof. Using the triangle inequality, one finds that

$$\begin{aligned} ||g(\boldsymbol{x}, \boldsymbol{y}) - g(\boldsymbol{u}, \boldsymbol{v})|| &= ||\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{u} - \boldsymbol{v}|| \\ &\leq ||\boldsymbol{x} - \boldsymbol{u}|| + ||\boldsymbol{y} - \boldsymbol{v}|| \\ &= ||(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{u}, \boldsymbol{v})||. \end{aligned}$$

In particular, g is Lipschitz continuous, hence also continuous. ■

Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space X.

3. The scalar multiplication $h(\lambda, \boldsymbol{x}) = \lambda \boldsymbol{x}$, where $\lambda \in \mathbb{F}$ and $\boldsymbol{x} \in X$.

Proof. To show that h is continuous at the point $(\lambda, \boldsymbol{x})$, let $\varepsilon > 0$ be given. Using the triangle inequality, one easily finds that

$$\begin{aligned} ||h(\lambda, \boldsymbol{x}) - h(\mu, \boldsymbol{y})|| &= ||\lambda \boldsymbol{x} - \lambda \boldsymbol{y} + \lambda \boldsymbol{y} - \mu \boldsymbol{y}|| \\ &\leq |\lambda| \cdot ||\boldsymbol{x} - \boldsymbol{y}|| + |\lambda - \mu| \cdot ||\boldsymbol{y}||. \end{aligned}$$

Suppose now that $||(\lambda, \boldsymbol{x}) - (\mu, \boldsymbol{y})|| < \delta$, where

$$\delta = \min \left\{ \frac{\varepsilon}{2|\lambda| + 1}, \frac{\varepsilon}{2||\boldsymbol{x}|| + 2}, 1 \right\}.$$

Then $|\lambda| \cdot ||\boldsymbol{x} - \boldsymbol{y}|| \leq \delta |\lambda| < \varepsilon/2$ and $||\boldsymbol{y}|| \leq \delta + ||\boldsymbol{x}|| \leq 1 + ||\boldsymbol{x}||$, so one also has $||h(\lambda, \boldsymbol{x}) - h(\mu, \boldsymbol{y})|| < \varepsilon/2 + \delta(1 + ||\boldsymbol{x}||) \leq \varepsilon$. ∎

Theorem 3.3 – Bounded means continuous

Suppose X, Y are normed vector spaces and let $T\colon X \to Y$ be linear. Then T is continuous if and only if T is bounded.

Proof. Suppose first that T is bounded. Then there exists a real number $M > 0$ such that $||T(\boldsymbol{x})|| \leq M||\boldsymbol{x}||$ for all $\boldsymbol{x} \in X$. Given any $\varepsilon > 0$, we may thus let $\delta = \varepsilon/M$ to conclude that

$$||\boldsymbol{x} - \boldsymbol{y}|| < \delta \quad \Longrightarrow \quad ||T(\boldsymbol{x}) - T(\boldsymbol{y})|| \leq M||\boldsymbol{x} - \boldsymbol{y}|| < \varepsilon.$$

Conversely, suppose T is continuous and let $\delta > 0$ be such that

$$||\boldsymbol{x} - \boldsymbol{y}|| < \delta \quad \Longrightarrow \quad ||T(\boldsymbol{x}) - T(\boldsymbol{y})|| < 1.$$

Given any nonzero $\boldsymbol{z} \in X$, we note that $\boldsymbol{x} = \frac{\delta \boldsymbol{z}}{2||\boldsymbol{z}||}$ satisfies $||\boldsymbol{x}|| < \delta$. This gives $||T(\boldsymbol{x})|| < 1$, so $||T(\boldsymbol{z})|| \leq \frac{2}{\delta}||\boldsymbol{z}||$ by linearity. ■

Theorem 3.4 – Norm of an operator

Suppose X, Y are normed vector spaces. Then the set $L(X,Y)$ of all bounded, linear operators $T\colon X \to Y$ is itself a normed vector space. In fact, one may define a norm on $L(X,Y)$ by letting

$$||T|| = \sup_{\boldsymbol{x}\neq 0} \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||}.$$

Proof, part 1. First, we check that $L(X,Y)$ is a vector space. Suppose that $T_1, T_2 \in L(X,Y)$ and let λ be a scalar. To see that the operators $T_1 + T_2$ and λT_1 are both bounded, we note that

$$\begin{aligned} ||T_1(\boldsymbol{x}) + T_2(\boldsymbol{x})|| &\leq ||T_1(\boldsymbol{x})|| + ||T_2(\boldsymbol{x})|| \leq C_1||\boldsymbol{x}|| + C_2||\boldsymbol{x}||, \\ ||\lambda T_1(\boldsymbol{x})|| &= |\lambda| \cdot ||T_1(\boldsymbol{x})|| \leq |\lambda| C_1 ||\boldsymbol{x}||. \end{aligned}$$

Since $T_1 + T_2$ and λT_1 are also linear, they are both in $L(X,Y)$.

Theorem 3.4 – Norm of an operator

Suppose X, Y are normed vector spaces. Then the set $L(X,Y)$ of all bounded, linear operators $T\colon X \to Y$ is itself a normed vector space. In fact, one may define a norm on $L(X,Y)$ by letting

$$||T|| = \sup_{\boldsymbol{x}\neq 0} \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||}.$$

Proof, part 2. We check that the given formula defines a norm. When $||T|| = 0$, we have $||T(\boldsymbol{x})|| = 0$ for all $\boldsymbol{x} \in X$ and this implies that T is the zero operator. Since $||\lambda T(\boldsymbol{x})|| = |\lambda| \cdot ||T(\boldsymbol{x})||$ for any scalar λ, it easily follows that $||\lambda T|| = |\lambda| \cdot ||T||$. Thus, it remains to prove the triangle inequality for the operator norm. Since

$$||S(\boldsymbol{x}) + T(\boldsymbol{x})|| \leq ||S(\boldsymbol{x})|| + ||T(\boldsymbol{x})|| \leq ||S|| \cdot ||\boldsymbol{x}|| + ||T|| \cdot ||\boldsymbol{x}||$$

for all $\boldsymbol{x} \neq 0$, we find that $||S + T|| \leq ||S|| + ||T||$, as needed. ∎

Theorem 3.5 – Euclidean norm

Suppose that X is a vector space with basis $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$. Then one may define a norm on X using the formula

$$\boldsymbol{x} = \sum_{i=1}^{k} c_i \boldsymbol{x}_i \quad \Longrightarrow \quad ||\boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{k} |c_i|^2}.$$

This norm is also known as the Euclidean or standard norm on X.

Proof. By definition, one has $||\boldsymbol{x}||_2 = ||\boldsymbol{c}||_2$. This is a norm on $\mathbb{R}^k$, so one may easily check that it is also a norm on X. For instance,

$$\begin{aligned} ||\boldsymbol{x}||_2 = 0 \quad \Longleftrightarrow \quad ||\boldsymbol{c}||_2 = 0 \quad &\Longleftrightarrow \quad \boldsymbol{c} = 0 \\ &\Longleftrightarrow \quad \boldsymbol{x} = 0. \end{aligned}$$

This proves the first property that a norm needs to satisfy, while the other two properties can be checked in a similar manner. ∎

Theorem 3.6 – Equivalence of all norms

The norms of a finite-dimensional vector space X are all equivalent.

Proof, part 1. Suppose $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ is a basis of X and let S denote the unit sphere in $\mathbb{R}^k$. Then the formula

$$f(c_1, c_2, \ldots, c_k) = \left\| \sum_{i=1}^{k} c_i \boldsymbol{x}_i \right\|$$

defines a continuous function $f\colon S \to \mathbb{R}$. Since S is closed and bounded in $\mathbb{R}^k$, it is also compact. Thus, f attains both a minimum value $\alpha > 0$ and a maximum value β. This gives

$$\alpha \leq \left\| \sum_{i=1}^{k} c_i \boldsymbol{x}_i \right\| \leq \beta$$

for every vector $\boldsymbol{c} \in \mathbb{R}^k$ which lies on the unit sphere S.

Theorem 3.6 – Equivalence of all norms

The norms of a finite-dimensional vector space X are all equivalent.

Proof, part 2. Suppose now that $\boldsymbol{x} \neq 0$ is arbitrary and write

$$\boldsymbol{x} = \sum_{i=1}^{k} d_i \boldsymbol{x}_i$$

for some coefficients $d_1, d_2, \ldots, d_k$. Then the norm $||\boldsymbol{d}||_2$ is nonzero and the vector $\boldsymbol{c} = \boldsymbol{d}/||\boldsymbol{d}||_2$ lies on the unit sphere, so we have

$$\alpha \leq \left|\left| \sum_{i=1}^{k} c_i \boldsymbol{x}_i \right|\right| \leq \beta.$$

Multiplying by $||\boldsymbol{d}||_2 = ||\boldsymbol{x}||_2$, we now get $\alpha||\boldsymbol{x}||_2 \leq ||\boldsymbol{x}|| \leq \beta||\boldsymbol{x}||_2$. Thus, the norm $||\boldsymbol{x}||$ is equivalent to the Euclidean norm $||\boldsymbol{x}||_2$. ∎

Theorem 3.7 – Examples of Banach spaces

1. Every finite-dimensional vector space X is a Banach space.

Proof. It suffices to prove completeness. Suppose $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ is a basis of X and let $\{\boldsymbol{y}_n\}$ be a Cauchy sequence in X. Expressing each $\boldsymbol{y}_n = \sum_{i=1}^k c_{ni}\boldsymbol{x}_i$ in terms of the basis, we find that

$$|c_{mi} - c_{ni}|^2 \leq \sum_{i=1}^k |c_{mi} - c_{ni}|^2 = ||\boldsymbol{y}_m - \boldsymbol{y}_n||_2^2,$$

so the sequence $\{c_{ni}\}$ is Cauchy for each i. Let c_i denote the limit of this sequence for each i and let $\boldsymbol{y} = \sum_{i=1}^k c_i\boldsymbol{x}_i$. Then we have

$$||\boldsymbol{y}_n - \boldsymbol{y}||_2^2 = \sum_{i=1}^k |c_{ni} - c_i|^2 \longrightarrow 0$$

as $n \to \infty$ and this implies that $\boldsymbol{y}_n$ converges to $\boldsymbol{y}$, as needed. ∎

Theorem 3.7 – Examples of Banach spaces

2. The sequence space ℓ^p is a Banach space for any $1 \leq p \leq \infty$.

Proof. We only treat the case $1 \leq p < \infty$ since the case $p = \infty$ is both similar and easier. Suppose $\{\boldsymbol{x}_n\}$ is a Cauchy sequence in ℓ^p and let $\varepsilon > 0$. Then there exists an integer N such that

$$|x_{mi} - x_{ni}|^p \leq \sum_{i=1}^{\infty} |x_{mi} - x_{ni}|^p = ||\boldsymbol{x}_m - \boldsymbol{x}_n||_p^p < \left(\frac{\varepsilon}{2}\right)^p$$

for all $m, n \geq N$. In particular, the sequence $\{x_{ni}\}_{n=1}^{\infty}$ is Cauchy for each i. Let x_i denote its limit. Given any $k \geq 1$, we then have

$$\sum_{i=1}^{k} |x_{mi} - x_{ni}|^p < \left(\frac{\varepsilon}{2}\right)^p \quad \Longrightarrow \quad \sum_{i=1}^{k} |x_i - x_{ni}|^p \leq \left(\frac{\varepsilon}{2}\right)^p.$$

It easily follows that $\boldsymbol{x}_n$ converges to $\boldsymbol{x}$ and that $\boldsymbol{x} \in \ell^p$. ■

Theorem 3.7 – Examples of Banach spaces

3. The space c_0 is a Banach space with respect to the $||\cdot||_\infty$ norm.

Proof. Suppose $\{\boldsymbol{x}_n\}$ is a Cauchy sequence in c_0. Since $c_0 \subset \ell^\infty$, this sequence must converge to an element $\boldsymbol{x} \in \ell^\infty$, so we need only show that the limit $\boldsymbol{x}$ is actually in c_0.

Let $\varepsilon > 0$ be given. Then there exists an integer N such that

$$||\boldsymbol{x}_n - \boldsymbol{x}||_\infty < \varepsilon/2 \quad \text{for all } n \geq N.$$

Since $\boldsymbol{x}_N \in c_0$, there also exists an integer N' such that $|x_{Nk}| < \varepsilon/2$ for all $k \geq N'$. In particular, one has

$$\begin{aligned} |x_k| &\leq |x_k - x_{Nk}| + |x_{Nk}| \\ &\leq ||\boldsymbol{x} - \boldsymbol{x}_N||_\infty + |x_{Nk}| < \varepsilon \end{aligned}$$

for all $k \geq N'$ and this implies that $\boldsymbol{x} \in c_0$, as needed. ∎

Theorem 3.7 – Examples of Banach spaces

4. If Y is a Banach space, then $L(X,Y)$ is a Banach space.

Proof. Suppose $\{T_n\}$ is a Cauchy sequence in $L(X,Y)$ and $\varepsilon > 0$. Then there exists an integer N such that

$$||T_n(\boldsymbol{x}) - T_m(\boldsymbol{x})|| \leq ||T_n - T_m|| \cdot ||\boldsymbol{x}|| \leq \frac{\varepsilon}{2}\,||\boldsymbol{x}||$$

for all $m, n \geq N$. Thus, the sequence $\{T_n(\boldsymbol{x})\}$ is also Cauchy. Let us denote its limit by $T(\boldsymbol{x})$. Then the map $\boldsymbol{x} \mapsto T(\boldsymbol{x})$ is linear and

$$||T_n(\boldsymbol{x}) - T(\boldsymbol{x})|| \leq \frac{\varepsilon}{2}\,||\boldsymbol{x}|| \quad \Longrightarrow \quad ||T_n - T|| < \varepsilon$$

for all $n \geq N$. This implies that T_n converges to T as $n \to \infty$ and that $T_N - T$ is bounded, so the operator T is bounded as well. ■

Theorem 3.8 – Absolute convergence implies convergence

Suppose that X is a Banach space and let $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ be a series which converges absolutely in X. Then this series must also converge.

Proof. Let $\varepsilon > 0$ be given and consider the partial sums

$$s_n = \sum_{i=1}^{n} \boldsymbol{x}_i, \qquad t_n = \sum_{i=1}^{n} ||\boldsymbol{x}_i||.$$

Since the sequence $\{t_n\}$ converges, it is also Cauchy. In particular, there exists an integer N such that $|t_m - t_n| < \varepsilon$ for all $m, n \geq N$. Assuming that $m > n \geq N$, we must then have

$$||s_m - s_n|| = \left|\left| \sum_{i=n+1}^{m} \boldsymbol{x}_i \right|\right| \leq \sum_{i=n+1}^{m} ||\boldsymbol{x}_i|| = t_m - t_n < \varepsilon.$$

Thus, $\{s_n\}$ is Cauchy as well, so it converges by completeness. ■

Theorem 3.9 – Geometric series

Suppose that $T\colon X \to X$ is a bounded linear operator on a Banach space X. If $||T|| < 1$, then $I - T$ is invertible with inverse $\sum_{n=0}^{\infty} T^n$.

Proof. The series $A = \sum_{n=0}^{\infty} T^n$ is absolutely convergent because

$$\sum_{n=0}^{\infty} ||T^n|| \leq \sum_{n=0}^{\infty} ||T||^n = \frac{1}{1 - ||T||}.$$

Since $L(X, X)$ is a Banach space, the given series is convergent as well. On the other hand, it is easy to check that

$$(I - T) \sum_{n=0}^{N} T^n = I - T^{N+1},$$

while $||T^{N+1}|| \leq ||T||^{N+1}$ goes to zero as $N \to \infty$. Taking the limit as $N \to \infty$, we may thus conclude that $(I - T)A = I$. ■

Theorem 3.10 – Set of invertible operators

Suppose X is a Banach space. Then the set of all invertible bounded linear operators $T\colon X \to X$ is an open subset of $L(X, X)$.

Proof. Let S be the set of all invertible operators $T \in L(X, X)$. To show that S is open in $L(X, X)$, we let $T \in S$ and we check that

$$||T - T'|| < \varepsilon \quad \Longrightarrow \quad T' \in S$$

when $\varepsilon = 1/||T^{-1}||$. Since the operator $A = T^{-1}(T - T')$ has norm

$$||A|| \leq ||T^{-1}|| \cdot ||T - T'|| < ||T^{-1}|| \cdot \varepsilon = 1,$$

the previous theorem ensures that $I - A$ is invertible. In particular,

$$I - A = I - T^{-1}(T - T') = T^{-1}T'$$

is invertible, so $T' = T(I - A)$ is also invertible and $T' \in S$. ■

Theorem 3.11 – Dual of $\mathbb{R}^k$

There is a bijective map $T\colon \mathbb{R}^k \to (\mathbb{R}^k)^*$ that sends each vector $\boldsymbol{a}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^k a_i x_i$.

Proof, part 1. The operator $T_{\boldsymbol{a}}$ is linear for each $\boldsymbol{a} \in \mathbb{R}^k$. To show that it is also bounded, we use Hölder's inequality to get

$$|T_{\boldsymbol{a}}(\boldsymbol{x})| \leq \sum_{i=1}^k |a_i| \cdot |x_i| \leq ||\boldsymbol{a}||_2 \cdot ||\boldsymbol{x}||_2.$$

This implies that $||T_{\boldsymbol{a}}|| \leq ||\boldsymbol{a}||_2$ for each $\boldsymbol{a} \in \mathbb{R}$. In particular, $T_{\boldsymbol{a}}$ is both bounded and linear, so it is an element of the dual $(\mathbb{R}^k)^*$.

Consider the map $T\colon \mathbb{R}^k \to (\mathbb{R}^k)^*$ which is defined by $T(\boldsymbol{a}) = T_{\boldsymbol{a}}$. To show it is injective, suppose that $T_{\boldsymbol{a}} = T_{\boldsymbol{b}}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$. Then $T_{\boldsymbol{a}}(\boldsymbol{e}_i) = T_{\boldsymbol{b}}(\boldsymbol{e}_i)$ for each standard unit vector $\boldsymbol{e}_i$, so $a_i = b_i$ for each i. In particular, $\boldsymbol{a} = \boldsymbol{b}$ and the given map is injective.

Theorem 3.11 – Dual of $\mathbb{R}^k$

There is a bijective map $T\colon \mathbb{R}^k \to (\mathbb{R}^k)^*$ that sends each vector $\boldsymbol{a}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^k a_i x_i$.

Proof, part 2. We now show that T is also surjective. Suppose S is an element of the dual $(\mathbb{R}^k)^*$ and consider the vector

$$\boldsymbol{a} = \big(S(\boldsymbol{e}_1), S(\boldsymbol{e}_2), \ldots, S(\boldsymbol{e}_k)\big) \in \mathbb{R}^k.$$

Given any $\boldsymbol{x} \in \mathbb{R}^k$, we can then write $\boldsymbol{x} = \sum_{i=1}^k x_i \boldsymbol{e}_i$ to find that

$$S(\boldsymbol{x}) = \sum_{i=1}^k x_i S(\boldsymbol{e}_i) = \sum_{i=1}^k a_i x_i = T_{\boldsymbol{a}}(\boldsymbol{x}).$$

This implies that $S = T_{\boldsymbol{a}}$, so the given map is also surjective. ∎

Theorem 3.12 – Dual of ℓ^p

Suppose $1 < p < \infty$ and let $q = p/(p-1)$. Then $1/p + 1/q = 1$ and there is a bijective map $T\colon \ell^q \to (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^{\infty} a_i x_i$.

Proof, part 1. The proof is very similar to the proof of the previous theorem. Given any sequences $\boldsymbol{a} \in \ell^q$ and $\boldsymbol{x} \in \ell^p$, one has

$$|T_{\boldsymbol{a}}(\boldsymbol{x})| \leq \sum_{i=1}^{\infty} |a_i| \cdot |x_i| \leq ||\boldsymbol{a}||_q \cdot ||\boldsymbol{x}||_p$$

by Hölder's inequality. This implies that $T_{\boldsymbol{a}}\colon \ell^p \to \mathbb{R}$ is a bounded linear operator, so it is an element of the dual $(\ell^p)^*$. One may thus define a map $T\colon \ell^q \to (\ell^p)^*$ by letting $T(\boldsymbol{a}) = T_{\boldsymbol{a}}$ for each $\boldsymbol{a} \in \ell^q$. Using the same argument as before, we can easily check that this map is injective. It remains to check that it is also surjective.

Theorem 3.12 – Dual of ℓ^p

Suppose $1 < p < \infty$ and let $q = p/(p-1)$. Then $1/p + 1/q = 1$ and there is a bijective map $T\colon \ell^q \to (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^{\infty} a_i x_i$.

Proof, part 2. Given an element S of the dual $(\ell^p)^*$, one may set

$$\boldsymbol{a} = \big(S(\boldsymbol{e}_1), S(\boldsymbol{e}_2), S(\boldsymbol{e}_3), \ldots\big)$$

and then proceed as before to conclude that $S = T_{\boldsymbol{a}}$. In this case, however, we also need to check that $\boldsymbol{a} \in \ell^q$. Consider the sequence

$$\boldsymbol{b} = \big(b_1, b_2, \ldots, b_n, 0, 0, \ldots\big), \qquad b_i = |a_i|^{q/p-1} a_i.$$

A simple computation gives $S(\boldsymbol{b}) = \sum_{i=1}^{n} |a_i|^q = ||\boldsymbol{b}||_p^p$ and this also implies that $||S|| \geq |S(\boldsymbol{b})|/||\boldsymbol{b}||_p = ||\boldsymbol{b}||_p^{p-1}$. Since the operator S is bounded, we conclude that $\boldsymbol{b} \in \ell^p$ and that $\boldsymbol{a} \in \ell^q$, as needed. ■