Chapter 3. Normed vector spaces
Proofs covered in class

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Theorem 3.1 – Product norm

Suppose $X, Y$ are normed vector spaces. Then one may define a norm on the product $X \times Y$ by letting $||(x, y)|| = ||x|| + ||y||$.

**Proof.** To see that the given formula defines a norm, we note that

$$ ||x|| + ||y|| = 0 \iff ||x|| = ||y|| = 0. $$

This implies that $||(x, y)|| = 0$ if and only if $x = y = 0$, while

$$ ||(\lambda x, \lambda y)|| = ||\lambda x|| + ||\lambda y|| = |\lambda| \cdot ||(x, y)||. $$

Adding the triangle inequalities in $X$ and $Y$, one also finds that

$$ ||(x_1, y_1) + (x_2, y_2)|| \leq ||(x_1, y_1)|| + ||(x_2, y_2)||. $$

In particular, the triangle inequality holds in $X \times Y$ as well. \qed
Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space $X$.

1. The norm $f(x) = ||x||$, where $x \in X$.

Proof. Using the triangle inequality, one finds that

$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y||,$$
$$||y|| = ||y - x + x|| \leq ||y - x|| + ||x||$$

for all $x, y \in X$. Rearranging terms now gives

$$|f(x) - f(y)| \leq ||x - y||$$

and this makes $f$ Lipschitz continuous, hence also continuous. ■
Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space $X$.

1. The vector addition $g(x, y) = x + y$, where $x, y \in X$.

Proof. Using the triangle inequality, one finds that

\[
\|g(x, y) - g(u, v)\| = \|x + y - u - v\| \\
\leq \|x - u\| + \|y - v\| \\
= \|(x, y) - (u, v)\|.
\]

In particular, $g$ is Lipschitz continuous, hence also continuous. ■
Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space $X$.

3. The scalar multiplication $h(\lambda, x) = \lambda x$, where $\lambda \in \mathbb{F}$ and $x \in X$.

Proof. To show that $h$ is continuous at the point $(\lambda, x)$, let $\varepsilon > 0$ be given. Using the triangle inequality, one easily finds that

$$
||h(\lambda, x) - h(\mu, y)|| = ||\lambda x - \lambda y + \lambda y - \mu y|| \\
\leq |\lambda| \cdot ||x - y|| + |\lambda - \mu| \cdot ||y||.
$$

Suppose now that $||(\lambda, x) - (\mu, y)|| < \delta$, where

$$
\delta = \min \left\{ \frac{\varepsilon}{2|\lambda| + 1}, \frac{\varepsilon}{2||x|| + 2}, 1 \right\}.
$$

Then $|\lambda| \cdot ||x - y|| \leq \delta|\lambda| < \varepsilon/2$ and $||y|| \leq \delta + ||x|| \leq 1 + ||x||$, so one also has $||h(\lambda, x) - h(\mu, y)|| < \varepsilon/2 + \delta(1 + ||x||) \leq \varepsilon$. $\blacksquare$
Theorem 3.3 – Bounded means continuous

Suppose $X, Y$ are normed vector spaces and let $T: X \to Y$ be linear. Then $T$ is continuous if and only if $T$ is bounded.

**Proof.** Suppose first that $T$ is bounded. Then there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. Given any $\varepsilon > 0$, we may thus let $\delta = \varepsilon/M$ to conclude that

$$\|x - y\| < \delta \implies \|T(x) - T(y)\| \leq M\|x - y\| < \varepsilon.$$ 

Conversely, suppose $T$ is continuous and let $\delta > 0$ be such that

$$\|x - y\| < \delta \implies \|T(x) - T(y)\| < 1.$$ 

Given any nonzero $z \in X$, we note that $x = \frac{\delta z}{2\|z\|}$ satisfies $\|x\| < \delta$. This gives $\|T(x)\| < 1$, so $\|T(z)\| \leq \frac{2}{\delta}\|z\|$ by linearity. ■
Theorem 3.4 – Norm of an operator

Suppose $X, Y$ are normed vector spaces. Then the set $L(X, Y)$ of all bounded, linear operators $T: X \rightarrow Y$ is itself a normed vector space. In fact, one may define a norm on $L(X, Y)$ by letting

$$||T|| = \sup_{x \neq 0} \frac{||T(x)||}{||x||}.$$ 

Proof, part 1. First, we check that $L(X, Y)$ is a vector space. Suppose that $T_1, T_2 \in L(X, Y)$ and let $\lambda$ be a scalar. To see that the operators $T_1 + T_2$ and $\lambda T_1$ are both bounded, we note that

$$||T_1(x) + T_2(x)|| \leq ||T_1(x)|| + ||T_2(x)|| \leq C_1||x|| + C_2||x||,$$

$$||\lambda T_1(x)|| = |\lambda| \cdot ||T_1(x)|| \leq |\lambda|C_1||x||.$$

Since $T_1 + T_2$ and $\lambda T_1$ are also linear, they are both in $L(X, Y)$.
**Theorem 3.4 – Norm of an operator**

Suppose $X, Y$ are normed vector spaces. Then the set $L(X, Y)$ of all bounded, linear operators $T: X \to Y$ is itself a normed vector space. In fact, one may define a norm on $L(X, Y)$ by letting

$$
||T|| = \sup_{x \neq 0} \frac{||T(x)||}{||x||}.
$$

**Proof, part 2.** We check that the given formula defines a norm. When $||T|| = 0$, we have $||T(x)|| = 0$ for all $x \in X$ and this implies that $T$ is the zero operator. Since $||\lambda T(x)|| = |\lambda| \cdot ||T(x)||$ for any scalar $\lambda$, it easily follows that $||\lambda T|| = |\lambda| \cdot ||T||$. Thus, it remains to prove the triangle inequality for the operator norm. Since

$$
||S(x) + T(x)|| \leq ||S(x)|| + ||T(x)|| \leq ||S|| \cdot ||x|| + ||T|| \cdot ||x||
$$

for all $x \neq 0$, we find that $||S + T|| \leq ||S|| + ||T||$, as needed. ■
Theorem 3.5 – Euclidean norm

Suppose that $X$ is a vector space with basis $x_1, x_2, \ldots, x_k$. Then one may define a norm on $X$ using the formula

$$x = \sum_{i=1}^{k} c_i x_i \implies ||x||_2 = \sqrt{\sum_{i=1}^{k} |c_i|^2}.$$

This norm is also known as the Euclidean or standard norm on $X$.

Proof. By definition, one has $||x||_2 = ||c||_2$. This is a norm on $\mathbb{R}^k$, so one may easily check that it is also a norm on $X$. For instance,

$$||x||_2 = 0 \iff ||c||_2 = 0 \iff c = 0 \iff x = 0.$$

This proves the first property that a norm needs to satisfy, while the other two properties can be checked in a similar manner.
Theorem 3.6 – Equivalence of all norms

The norms of a finite-dimensional vector space \( X \) are all equivalent.

**Proof, part 1.** Suppose \( x_1, x_2, \ldots, x_k \) is a basis of \( X \) and let \( S \) denote the unit sphere in \( \mathbb{R}^k \). Then the formula

\[
f(c_1, c_2, \ldots, c_k) = \left\| \sum_{i=1}^{k} c_i x_i \right\|
\]

defines a continuous function \( f: S \to \mathbb{R} \). Since \( S \) is closed and bounded in \( \mathbb{R}^k \), it is also compact. Thus, \( f \) attains both a minimum value \( \alpha > 0 \) and a maximum value \( \beta \). This gives

\[
\alpha \leq \left\| \sum_{i=1}^{k} c_i x_i \right\| \leq \beta
\]

for every vector \( c \in \mathbb{R}^k \) which lies on the unit sphere \( S \).
Theorem 3.6 – Equivalence of all norms

The norms of a finite-dimensional vector space $X$ are all equivalent.

**Proof, part 2.** Suppose now that $x \neq 0$ is arbitrary and write

$$x = \sum_{i=1}^{k} d_i x_i$$

for some coefficients $d_1, d_2, \ldots, d_k$. Then the norm $||d||_2$ is nonzero and the vector $c = d/||d||_2$ lies on the unit sphere, so we have

$$\alpha \leq \left| \left| \sum_{i=1}^{k} c_i x_i \right| \right| \leq \beta.$$

Multiplying by $||d||_2 = ||x||_2$, we now get $\alpha ||x||_2 \leq ||x|| \leq \beta ||x||_2$. Thus, the norm $||x||$ is equivalent to the Euclidean norm $||x||_2$. □
Every finite-dimensional vector space \( X \) is a Banach space.

**Proof.** It suffices to prove completeness. Suppose \( x_1, x_2, \ldots, x_k \) is a basis of \( X \) and let \( \{y_n\} \) be a Cauchy sequence in \( X \). Expressing each \( y_n = \sum_{i=1}^{k} c_{ni} x_i \) in terms of the basis, we find that

\[
|c_{mi} - c_{ni}|^2 \leq \sum_{i=1}^{k} |c_{mi} - c_{ni}|^2 = \|y_m - y_n\|_2^2,
\]

so the sequence \( \{c_{ni}\} \) is Cauchy for each \( i \). Let \( c_i \) denote the limit of this sequence for each \( i \) and let \( y = \sum_{i=1}^{k} c_i x_i \). Then we have

\[
\|y_n - y\|_2^2 = \sum_{i=1}^{k} |c_{ni} - c_i|^2 \longrightarrow 0
\]

as \( n \to \infty \) and this implies that \( y_n \) converges to \( y \), as needed.
The sequence space $\ell^p$ is a Banach space for any $1 \leq p \leq \infty$.

**Proof.** We only treat the case $1 \leq p < \infty$ since the case $p = \infty$ is both similar and easier. Suppose $\{x_n\}$ is a Cauchy sequence in $\ell^p$ and let $\varepsilon > 0$. Then there exists an integer $N$ such that

$$|x_{mi} - x_{ni}|^p \leq \sum_{i=1}^{\infty} |x_{mi} - x_{ni}|^p = \|x_m - x_n\|_p^p < \left(\frac{\varepsilon}{2}\right)^p$$

for all $m, n \geq N$. In particular, the sequence $\{x_{ni}\}_{n=1}^{\infty}$ is Cauchy for each $i$. Let $x_i$ denote its limit. Given any $k \geq 1$, we then have

$$\sum_{i=1}^{k} |x_{mi} - x_{ni}|^p < \left(\frac{\varepsilon}{2}\right)^p \implies \sum_{i=1}^{k} |x_{i} - x_{ni}|^p \leq \left(\frac{\varepsilon}{2}\right)^p.$$

It easily follows that $x_n$ converges to $x$ and that $x \in \ell^p$. \qed
Theorem 3.7 – Examples of Banach spaces

3 The space $c_0$ is a Banach space with respect to the $\| \cdot \|_\infty$ norm.

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in $c_0$. Since $c_0 \subset \ell^\infty$, this sequence must converge to an element $x \in \ell^\infty$, so we need only show that the limit $x$ is actually in $c_0$.

Let $\varepsilon > 0$ be given. Then there exists an integer $N$ such that

$$\|x_n - x\|_\infty < \varepsilon/2 \quad \text{for all } n \geq N.$$ 

Since $x_N \in c_0$, there also exists an integer $N'$ such that $|x_{Nk}| < \varepsilon/2$ for all $k \geq N'$. In particular, one has

$$|x_k| \leq |x_k - x_{Nk}| + |x_{Nk}|$$

$$\leq \|x - x_N\|_\infty + |x_{Nk}| < \varepsilon$$

for all $k \geq N'$ and this implies that $x \in c_0$, as needed.
Theorem 3.7 – Examples of Banach spaces

If $Y$ is a Banach space, then $L(X, Y)$ is a Banach space.

**Proof.** Suppose $\{T_n\}$ is a Cauchy sequence in $L(X, Y)$ and $\varepsilon > 0$. Then there exists an integer $N$ such that

$$||T_n(x) - T_m(x)|| \leq ||T_n - T_m|| \cdot ||x|| \leq \frac{\varepsilon}{2} ||x||$$

for all $m, n \geq N$. Thus, the sequence $\{T_n(x)\}$ is also Cauchy. Let us denote its limit by $T(x)$. Then the map $x \mapsto T(x)$ is linear and

$$||T_n(x) - T(x)|| \leq \frac{\varepsilon}{2} ||x|| \quad \Longrightarrow \quad ||T_n - T|| < \varepsilon$$

for all $n \geq N$. This implies that $T_n$ converges to $T$ as $n \to \infty$ and that $T_N - T$ is bounded, so the operator $T$ is bounded as well. ■
Theorem 3.8 – Absolute convergence implies convergence

Suppose that \( X \) is a Banach space and let \( \sum_{n=1}^{\infty} x_n \) be a series which converges absolutely in \( X \). Then this series must also converge.

**Proof.** Let \( \varepsilon > 0 \) be given and consider the partial sums

\[
 s_n = \sum_{i=1}^{n} x_i, \quad t_n = \sum_{i=1}^{n} \|x_i\|.
\]

Since the sequence \( \{t_n\} \) converges, it is also Cauchy. In particular, there exists an integer \( N \) such that \( |t_m - t_n| < \varepsilon \) for all \( m, n \geq N \). Assuming that \( m > n \geq N \), we must then have

\[
 \|s_m - s_n\| = \left\| \sum_{i=n+1}^{m} x_i \right\| \leq \sum_{i=n+1}^{m} \|x_i\| = t_m - t_n < \varepsilon.
\]

Thus, \( \{s_n\} \) is Cauchy as well, so it converges by completeness. \( \blacksquare \)
Theorem 3.9 – Geometric series

Suppose that $T : X \to X$ is a bounded linear operator on a Banach space $X$. If $\|T\| < 1$, then $I - T$ is invertible with inverse $\sum_{n=0}^{\infty} T^n$.

**Proof.** The series $A = \sum_{n=0}^{\infty} T^n$ is absolutely convergent because

$$\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|}. $$

Since $L(X, X)$ is a Banach space, the given series is convergent as well. On the other hand, it is easy to check that

$$(I - T) \sum_{n=0}^{N} T^n = I - T^{N+1},$$

while $\|T^{N+1}\| \leq \|T\|^{N+1}$ goes to zero as $N \to \infty$. Taking the limit as $N \to \infty$, we may thus conclude that $(I - T)A = I$.  

$\blacksquare$
**Theorem 3.10 – Set of invertible operators**

Suppose $X$ is a Banach space. Then the set of all invertible bounded linear operators $T: X \to X$ is an open subset of $L(X, X)$.

**Proof.** Let $S$ be the set of all invertible operators $T \in L(X, X)$. To show that $S$ is open in $L(X, X)$, we let $T \in S$ and we check that

$$||T - T'|| < \varepsilon \implies T' \in S$$

when $\varepsilon = 1/||T^{-1}||$. Since the operator $A = T^{-1}(T - T')$ has norm

$$||A|| \leq ||T^{-1}|| \cdot ||T - T'|| < ||T^{-1}|| \cdot \varepsilon = 1,$$

the previous theorem ensures that $I - A$ is invertible. In particular,

$$I - A = I - T^{-1}(T - T') = T^{-1}T'$$

is invertible, so $T' = T(I - A)$ is also invertible and $T' \in S$. ■
Theorem 3.11 – Dual of $\mathbb{R}^k$

There is a bijective map $T: \mathbb{R}^k \to (\mathbb{R}^k)^*$ that sends each vector $a$ to the bounded linear operator $T_a$ defined by $T_a(x) = \sum_{i=1}^{k} a_i x_i$.

Proof, part 1. The operator $T_a$ is linear for each $a \in \mathbb{R}^k$. To show that it is also bounded, we use Hölder’s inequality to get

$$|T_a(x)| \leq \sum_{i=1}^{k} |a_i| \cdot |x_i| \leq ||a||_2 \cdot ||x||_2.$$  

This implies that $||T_a|| \leq ||a||_2$ for each $a \in \mathbb{R}$. In particular, $T_a$ is both bounded and linear, so it is an element of the dual $(\mathbb{R}^k)^*$.

Consider the map $T: \mathbb{R}^k \to (\mathbb{R}^k)^*$ which is defined by $T(a) = T_a$. To show it is injective, suppose that $T_a = T_b$ for some $a, b \in \mathbb{R}^k$. Then $T_a(e_i) = T_b(e_i)$ for each standard unit vector $e_i$, so $a_i = b_i$ for each $i$. In particular, $a = b$ and the given map is injective.
Theorem 3.11 – Dual of \( \mathbb{R}^k \)

There is a bijective map \( T: \mathbb{R}^k \to (\mathbb{R}^k)^* \) that sends each vector \( a \) to the bounded linear operator \( T_a \) defined by \( T_a(x) = \sum_{i=1}^{k} a_i x_i \).

**Proof, part 2.** We now show that \( T \) is also surjective. Suppose \( S \) is an element of the dual \( (\mathbb{R}^k)^* \) and consider the vector

\[
a = (S(e_1), S(e_2), \ldots, S(e_k)) \in \mathbb{R}^k.
\]

Given any \( x \in \mathbb{R}^k \), we can then write \( x = \sum_{i=1}^{k} x_i e_i \) to find that

\[
S(x) = \sum_{i=1}^{k} x_i S(e_i) = \sum_{i=1}^{k} a_i x_i = T_a(x).
\]

This implies that \( S = T_a \), so the given map is also surjective. \( \blacksquare \)
Theorem 3.12 – Dual of $\ell^p$

Suppose $1 < p < \infty$ and let $q = \frac{p}{(p - 1)}$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and there is a bijective map $T: \ell^q \rightarrow (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_a$ defined by $T_a(x) = \sum_{i=1}^{\infty} a_i x_i$.

**Proof, part 1.** The proof is very similar to the proof of the previous theorem. Given any sequences $a \in \ell^q$ and $x \in \ell^p$, one has

$$|T_a(x)| \leq \sum_{i=1}^{\infty} |a_i| \cdot |x_i| \leq \|a\|_q \cdot \|x\|_p$$

by Hölder’s inequality. This implies that $T_a: \ell^p \rightarrow \mathbb{R}$ is a bounded linear operator, so it is an element of the dual $(\ell^p)^*$. One may thus define a map $T: \ell^q \rightarrow (\ell^p)^*$ by letting $T(a) = T_a$ for each $a \in \ell^q$. Using the same argument as before, we can easily check that this map is injective. It remains to check that it is also surjective.
Theorem 3.12 – Dual of $\ell^p$

Suppose $1 < p < \infty$ and let $q = \frac{p}{p - 1}$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and there is a bijective map $T: \ell^q \to (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_a$ defined by $T_a(x) = \sum_{i=1}^{\infty} a_i x_i$.

Proof, part 2. Given an element $S$ of the dual $(\ell^p)^*$, one may set

$$ a = (S(e_1), S(e_2), S(e_3), \ldots) $$

and then proceed as before to conclude that $S = T_a$. In this case, however, we also need to check that $a \in \ell^q$. Consider the sequence

$$ b = (b_1, b_2, \ldots, b_n, 0, 0, \ldots), \quad b_i = |a_i|^{q/p - 1} a_i. $$

A simple computation gives $S(b) = \sum_{i=1}^{n} |a_i|^q = \|b\|_p^p$ and this also implies that $\|S\| \geq \|S(b)\| / \|b\|_p \geq \|b\|_p^{p-1}$. Since the operator $S$ is bounded, we conclude that $b \in \ell^p$ and that $a \in \ell^q$, as needed.