Chapter 2. Topological spaces

Proofs covered in class

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Theorem 2.1 – Limits are not necessarily unique

Suppose that X has the indiscrete topology and let $x \in X$. Then the constant sequence $x_n = x$ converges to y for every $y \in X$.

Proof. Suppose U is an open set that contains y. Since X has the indiscrete topology, the only open sets are $\varnothing$ and X, so U must be equal to X. This implies that $x_n \in U$ for all $n \geq 1$. In view of the definition of convergence, we thus have $x_n \to y$ as $n \to \infty$. ■

Theorem 2.2 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.
2. Both $\varnothing$ and X are closed in X.

Proof. First, we prove 1. Suppose $\{x_n\}$ is a convergent sequence of points of A and let x denote its limit. To show that $x \in A$, we assume $x \in X - A$ for the sake of contradiction. Then $X - A$ is an open set which contains the limit x, so there is an integer N such that $x_n \in X - A$ for all $n \geq N$. This contradicts our assumption that $x_n \in A$ for all $n \geq 1$. Thus, we must have $x \in A$.

Next, we turn to 2. By definition, the sets $\varnothing, X$ are both open in X, so their complements $X, \varnothing$ are both closed in X. ∎

Theorem 2.2 – Main facts about closed sets

3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

Proof. We prove these statements using De Morgan's laws

$$X - \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (X - U_i), \qquad X - \bigcap_{i} U_i = \bigcup_{i} (X - U_i).$$

To prove 3, suppose that the sets U_i are closed in X. Then their complements $X - U_i$ are open in X and these are finitely many, so their intersection is open in X. Using the first De Morgan's law, we conclude that the union of the sets U_i is closed in X.

The proof of 4 is similar. If the sets U_i are closed in X, then their complements $X - U_i$ are open in X and so is their union. Using the second De Morgan's law, we conclude that $\bigcap U_i$ is closed in X. ■

Theorem 2.3 – Main facts about the closure

1. One has $A \subset \overline{A}$ for any set A.
2. If $A \subset B$, then $\overline{A} \subset \overline{B}$ as well.
3. The set A is closed if and only if $\overline{A} = A$.
4. The closure of $\overline{A}$ is itself, namely $\overline{\overline{A}} = \overline{A}$.

Proof. By definition, $\overline{A}$ is the smallest closed set that contains A, so 1 is clear. To prove 2, suppose $A \subset B$. Then $A \subset B \subset \overline{B}$ and so $\overline{B}$ is a closed set that contains A. Since $\overline{A}$ is the smallest such closed set by definition, we conclude that $\overline{A} \subset \overline{B}$.

Part 3 should be clear because $\overline{A}$ is the smallest closed set that contains A. In particular, $\overline{A}$ is equal to A if and only if A is closed. Finally, part 4 is a direct consequence of part 3. Since $\overline{A}$ is a closed set by definition, it must be equal to its own closure. ■

Theorem 2.4 – Main facts about the interior

1. One has $A^\circ \subset A$ for any set A.
2. If $A \subset B$, then $A^\circ \subset B^\circ$ as well.
3. The set A is open if and only if $A^\circ = A$.
4. The interior of A° is itself, namely $(A^\circ)^\circ = A^\circ$.

Proof. By definition, A° is the largest open set contained in A, so part 1 is clear. To prove 2, suppose $A \subset B$. Then $A^\circ \subset A \subset B$ and this makes A° an open set which is contained in B. Since B° is the largest such open set by definition, we conclude that $A^\circ \subset B^\circ$.

Part 3 should be clear since A° is the largest open set contained in A. In particular, A° is equal to A if and only if A is open. Finally, part 4 is a direct consequence of part 3. Since A° is an open set by definition, it must be equal to its own interior. ∎

Theorem 2.5 – Characterisation of closure/interior/boundary

Suppose (X, T) is a topological space and let $A \subset X$.

1. $x \in \overline{A} \iff$ every neighbourhood of x intersects A.

Proof. By definition, the closure $\overline{A}$ is the intersection of all closed sets that contain A. In other words, we have

$$x \notin \overline{A} \iff x \notin C \text{ for some closed set } C \text{ that contains } A.$$

Setting $U = X - C$ for convenience, we conclude that

$$\begin{aligned} x \notin \overline{A} &\iff x \in U \text{ for some open set } U \text{ contained in } X - A \\ &\iff \text{some neighbourhood of } x \text{ is contained in } X - A \\ &\iff \text{some neighbourhood of } x \text{ does not intersect } A. \end{aligned}$$

This is precisely the statement of the theorem. ∎

Theorem 2.5 – Characterisation of closure/interior/boundary

Suppose (X, T) is a topological space and let $A \subset X$.

2. $x \in A^\circ \iff$ some neighbourhood of x lies within A.
3. $x \in \partial A \iff$ every neighbourhood of x intersects A and $X - A$.

Proof. By definition, the interior A° is the union of all open sets which are contained in A. Thus, we have

$$\begin{aligned} x \in A^\circ \quad &\iff \quad x \in U \text{ for some open set } U \text{ contained in } A \\ &\iff \quad \text{some neighbourhood of } x \text{ is contained in } A. \end{aligned}$$

This settles part 2. To prove 3, we recall 1 which states that

$$x \in \overline{A} \quad \iff \quad \text{every neighbourhood of } x \text{ intersects } A.$$

Since $\partial A = \overline{A} \cap \overline{X - A}$ by definition, the result now follows. ∎

Theorem 2.6 – Interior, closure and boundary

One has $A^\circ \cap \partial A = \varnothing$ and also $A^\circ \cup \partial A = \overline{A}$ for any set A.

Proof. If $x \in A^\circ$, then x has a neighbourhood U that lies within A and this neighbourhood does not intersect $X - A$, so $x \notin \partial A$. This proves the first statement. To prove the second, we recall that

$$A^\circ \subset A \subset \overline{A} \quad \text{and} \quad \partial A \subset \overline{A}.$$

This implies the inclusion $A^\circ \cup \partial A \subset \overline{A}$, so it remains to prove the opposite inclusion. Suppose that $x \in \overline{A}$. Then every neighbourhood of x intersects A and we examine two cases. If every neighbourhood of x intersects $X - A$, then we must have $x \in \partial A$. Otherwise, there is a neighbourhood of x that does not intersect $X - A$. Since this neighbourhood lies entirely within A, we must have $x \in A^\circ$. ■

Theorem 2.7 – Limit points and closure

Let (X, T) be a topological space and let $A \subset X$. If A' is the set of all limit points of A, then the closure of A is $\overline{A} = A \cup A'$.

Proof. One has $A \subset \overline{A}$ by definition. To see that $A' \subset \overline{A}$ as well, suppose that $x \in A'$. Then every neighbourhood of x intersects A at a point other than x, so $x \in \overline{A}$. This proves the inclusion

$$A \cup A' \subset \overline{A}.$$

To prove the opposite inclusion, suppose that $x \in \overline{A}$, but $x \notin A$. Then every neighbourhood of x intersects A at a point other than x and so $x \in A'$. This proves the opposite inclusion $\overline{A} \subset A \cup A'$. ■

Theorem 2.8 – Composition of continuous functions

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between topological spaces. Then the composition $g \circ f\colon X \to Z$ is continuous.

Proof. Suppose that U is open in Z. Then $g^{-1}(U)$ is open in Y by continuity and similarly $f^{-1}(g^{-1}(U))$ is open in X. On the other hand, it is easy to check that

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= \{x \in X : f(x) \in g^{-1}(U)\} \\ &= \{x \in X : g(f(x)) \in U\} \\ &= (g \circ f)^{-1}(U). \end{aligned}$$

Thus, $(g \circ f)^{-1}(U)$ is open in X and so $g \circ f$ is continuous. ■

Theorem 2.9 – Continuity and sequences

Let $f\colon X \to Y$ be a continuous function between topological spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Proof. Let U be an open set that contains $f(x)$. Then $f^{-1}(U)$ is an open set that contains x. Since $x_n \to x$ as $n \to \infty$, there is an integer N such that $x_n \in f^{-1}(U)$ for all $n \geq N$. Thus, $f(x_n) \in U$ for all $n \geq N$ and this means that $f(x_n)$ converges to $f(x)$. ■

Theorem 2.10 – Inclusion maps are continuous

Let (X, T) be a topological space and let $A \subset X$. Then the inclusion map $i\colon A \to X$ defined by $i(x) = x$ is continuous.

Proof. Suppose U is an open set in X. Its inverse image is then

$$\begin{aligned} i^{-1}(U) &= \{x \in A : i(x) \in U\} \\ &= \{x \in A : x \in U\} \quad = A \cap U. \end{aligned}$$

Since U is open in X, the intersection $A \cap U$ is open in A by the definition of the subspace topology. Thus, i is continuous. ■

Theorem 2.11 – Restriction maps are continuous

Let $f\colon X \to Y$ be a continuous function between topological spaces and let $A \subset X$. Then the restriction map $g\colon A \to Y$ which is defined by $g(x) = f(x)$ is continuous. This map is often denoted by $g = f|_A$.

Proof. Suppose U is an open set in Y. Its inverse image is then

$$\begin{aligned} g^{-1}(U) &= \{x \in A : g(x) \in U\} \\ &= \{x \in A : f(x) \in U\} = A \cap f^{-1}(U). \end{aligned}$$

By continuity, $f^{-1}(U)$ is open in X, so $A \cap f^{-1}(U)$ is open in A. Thus, $g^{-1}(U)$ is open in A and the function g is continuous. ∎

Theorem 2.12 – Projection maps are continuous

Let (X, T) and (Y, T') be topological spaces. If $X \times Y$ is equipped with the product topology, then the projection map $p_1 \colon X \times Y \to X$ defined by $p_1(x, y) = x$ is continuous. Moreover, the same is true for the projection map $p_2 \colon X \times Y \to Y$ defined by $p_2(x, y) = y$.

Proof. Given a set U which is open in X, one easily finds that

$$\begin{aligned} p_1^{-1}(U) &= \{(x, y) \in X \times Y : p_1(x, y) \in U\} \\ &= \{(x, y) \in X \times Y : x \in U\} \\ &= U \times Y. \end{aligned}$$

Since this is open in the product topology of $X \times Y$, the projection map p_1 is continuous. Similarly, one has $p_2^{-1}(V) = X \times V$ for each set V which is open in Y, so p_2 is continuous as well. ■

Theorem 2.13 – Continuous map into a product space

Let X, Y, Z be topological spaces. Then a function $f\colon Z \to X \times Y$ is continuous if and only if its components $p_1 \circ f$, $p_2 \circ f$ are continuous.

Proof. First, suppose f is continuous. Then $p_1 \circ f$ and $p_2 \circ f$ are compositions of continuous functions, so they are both continuous.

Conversely, suppose $p_1 \circ f$ and $p_2 \circ f$ are both continuous. To show that f is continuous, it suffices to show that $f^{-1}(U \times V)$ is open in Z whenever U is open in X and V is open in Y. Since

$$\begin{aligned} f^{-1}(U \times V) &= \{z \in Z : f(z) \in U \times V\} \\ &= \{z \in Z : p_1(f(z)) \in U \text{ and } p_2(f(z)) \in V\} \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V), \end{aligned}$$

we find that $f^{-1}(U \times V)$ is open in Z and so f is continuous. ∎

Theorem 2.14 – Main facts about Hausdorff spaces

1. Every metric space is Hausdorff.

Proof. Let $x \neq y$ be points of a metric space X. Then $r = d(x, y)$ is positive and we shall consider the open sets

$$U = B(x, r/2), \qquad V = B(y, r/2).$$

Since $x \in U$ and $y \in V$, it remains to show that $U \cap V$ is empty. Suppose then that $z \in U \cap V$. Then we must have

$$r = d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2,$$

a contradiction. Thus, $U \cap V$ is empty and X is Hausdorff. ■

Theorem 2.14 – Main facts about Hausdorff spaces

2. Every subset of a Hausdorff space is Hausdorff.

Proof. Suppose that X is Hausdorff and let $A \subset X$. Given any two points $x \neq y$ in A, we can find disjoint sets U_x, U_y which are open in X with $x \in U_x$ and $y \in U_y$. Intersecting these sets with A, we find that $U_x \cap A$ and $U_y \cap A$ are open in A with

$$x \in U_x \cap A, \qquad y \in U_y \cap A, \qquad (U_x \cap A) \cap (U_y \cap A) = \varnothing.$$

This shows that the subset A is Hausdorff as well. ∎

Theorem 2.14 – Main facts about Hausdorff spaces

3. Every finite subset of a Hausdorff space is closed.

Proof. Suppose X is Hausdorff. If we can show that the set $\{x\}$ is closed for each $x \in X$, then this will imply that every finite set is a finite union of closed sets, hence also closed.

It remains to show that $\{x\}$ is closed for each $x \in X$. Given any point $y \neq x$, we can find open sets U_x, U_y with

$$x \in U_x, \qquad y \in U_y, \qquad U_x \cap U_y = \varnothing.$$

In particular, each point $y \neq x$ has a neighbourhood that does not intersect $\{x\}$ and so y is not in the closure of $\{x\}$. This means that the closure of $\{x\}$ is $\{x\}$ itself, so this set is closed, indeed. ∎

Theorem 2.14 – Main facts about Hausdorff spaces

4. The product of two Hausdorff spaces is Hausdorff.

Proof. Suppose X, Y are both Hausdorff. To show that $X \times Y$ is Hausdorff as well, let (x_1, y_1) and (x_2, y_2) be two distinct points. Then we have either $x_1 \neq x_2$ or else $y_1 \neq y_2$.

Suppose that $x_1 \neq x_2$, as the other case is similar. Then there exist sets U_{x_1}, U_{x_2} which are open in X with

$$x_1 \in U_{x_1}, \qquad x_2 \in U_{x_2}, \qquad U_{x_1} \cap U_{x_2} = \varnothing.$$

Then $V_1 = U_{x_1} \times Y$ and $V_2 = U_{x_2} \times Y$ are open in $X \times Y$ with

$$(x_1, y_1) \in V_1, \qquad (x_2, y_2) \in V_2, \qquad V_1 \cap V_2 = \varnothing.$$

This shows that the product $X \times Y$ is Hausdorff as well. ■

Theorem 2.14 – Main facts about Hausdorff spaces

5. A convergent sequence in a Hausdorff space has a unique limit.

Proof. Suppose X is Hausdorff and let $\{x_n\}$ be a sequence that has two different limits $x \neq y$. Then there exist open sets U_x, U_y with

$$x \in U_x, \qquad y \in U_y, \qquad U_x \cap U_y = \varnothing.$$

Since the limit x lies in U_x, there is an integer N_1 such that

$$x_n \in U_x \quad \text{for all } n \geq N_1.$$

Since the limit y lies in U_y, there is an integer N_2 such that

$$x_n \in U_y \quad \text{for all } n \geq N_2.$$

This actually gives $x_n \in U_x \cap U_y$ for all $n \geq \max\{N_1, N_2\}$, which is contrary to the fact that the intersection $U_x \cap U_y$ is empty. ∎

Theorem 2.15 – Some facts about connected spaces

1. To say that X is connected is to say that the only subsets of X which are both open and closed in X are the subsets $\varnothing, X$.

Proof. Suppose A is both open and closed in X, but A is neither empty nor equal to X. Then A and $B = X - A$ are nonempty, open and disjoint with $A \cup B = X$, so they form a partition of X.

Conversely, suppose A, B form a partition of X. Then A, B are nonempty, open and disjoint with $A \cup B = X$. This means that A is neither empty nor equal to X. On the other hand, $B = X - A$ is open in X, so A is both open and closed in X. ∎

Theorem 2.15 – Some facts about connected spaces

2. The continuous image of a connected space is connected: if X is connected and $f\colon X \to Y$ is continuous, then $f(X)$ is connected.

Proof. We may assume that $f(X) = Y$ without loss of generality. Suppose that A, B form a partition of Y. Then A, B are nonempty, open and disjoint with $A \cup B = Y$. Since f is continuous, it easily follows that the inverse images

$$U = f^{-1}(A), \qquad V = f^{-1}(B)$$

are nonempty, open and disjoint with $U \cup V = X$. In other words, they form a partition of X, which is contrary to the fact that X is connected. This implies that Y must be connected as well. ■

Theorem 2.15 – Some facts about connected spaces

3. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof, part 1. An interval is a set $I \subset \mathbb{R}$ which contains all points between $\inf I$ and $\sup I$. Suppose that $A \subset \mathbb{R}$ is a set which is not an interval. Then there exists some real number x such that

$$\inf A < x < \sup A, \qquad x \notin A.$$

Since x is larger than the greatest lower bound of A, we see that x is not a lower bound of A, so $a < x$ for some $a \in A$. Similarly, we must also have $x < b$ for some $b \in A$. It easily follows that the sets

$$U = (-\infty, x) \cap A, \qquad V = (x, +\infty) \cap A$$

are nonempty, disjoint and open in A with $U \cup V = A$. Thus, these sets form a partition of A and so A is not connected.

Theorem 2.15 – Some facts about connected spaces

3. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof, part 2. Conversely, suppose that $I \subset \mathbb{R}$ is an interval which has a partition $U|V$. Pick two points $x \in U$ and $y \in V$. Assuming that $x < y$ without loss of generality, we now set

$$U' = [x, y] \cap U, \qquad z = \sup\, U'.$$

Given any integer $n \in \mathbb{N}$, there exists a point $x_n \in U'$ such that

$$z - 1/n \leq x_n \leq z.$$

Since U' is closed in U and $x_n \to z$ as $n \to \infty$, we see that $z \in U'$. In particular, $z \in U$ and $x \leq z < y$. Since U is open in I, we must have $z + \varepsilon \in U$ for all small enough $\varepsilon > 0$ and so $z + \varepsilon \in U'$ for all small enough $\varepsilon > 0$. This contradicts the fact that $z = \sup U'$. ∎

Theorem 2.15 – Some facts about connected spaces

4. If a connected space A is a subset of X and the sets U, V form a partition of X, then A must lie entirely within either U or V.

Proof. Consider the sets $A \cap U$ and $A \cap V$. These are open in A, they are disjoint and their union is equal to

$$(A \cap U) \cup (A \cap V) = A \cap (U \cup V) = A \cap X = A.$$

Since A is connected, one of the two sets must be empty. Suppose that $A \cap U$ is empty, as the other case is similar. Then A is a subset of $X = U \cup V$ which does not intersect U, so $A \subset V$. ∎

Theorem 2.16 – Some more facts about connected spaces

1. If A is a connected subset of X, then $\overline{A}$ is connected as well.

Proof. Suppose U, V form a partition of the closure $\overline{A}$. Since A is a connected subset of this partition, it must lie within either U or V. Assume that $A \subset U$ without loss of generality. Then we have

$$A \subset U \subset X - V$$

and this makes $X - V$ a closed set that contains A. In view of the definition of the closure, the smallest such set is $\overline{A}$, so

$$\overline{A} \subset X - V \quad \Longrightarrow \quad U \cup V \subset X - V.$$

This means that V must be empty, which is contrary to assumption. In particular, $\overline{A}$ has no partition and the result follows. ■

Theorem 2.16 – Some more facts about connected spaces

2. Consider a collection of connected sets U_i that have a point in common. Then the union of these sets is connected as well.

Proof. Suppose A, B form a partition of the union and let x be a point which is contained in U_i for all i. Then x belongs to either A or B. Assume that $x \in A$ without loss of generality. Since U_i is a connected subset of the partition, it must lie entirely within either A or B. Since U_i contains x, however, we must have

$$U_i \subset A \text{ for all } i \quad \Longrightarrow \quad \bigcup_i U_i \subset A.$$

This means that B must be empty, which is contrary to assumption. In particular, $\bigcup_i U_i$ has no partition and the result follows. ■

Theorem 2.16 – Some more facts about connected spaces

3. The product of two connected spaces is connected.

Proof. Suppose X, Y are connected spaces and let $(x, y) \in X \times Y$. The set $X \times \{y\}$ corresponds to a horizontal line in $X \times Y$ and it is also the image of X under the function

$$f\colon X \to X \times \{y\}, \qquad f(x) = (x, y).$$

Since X is connected and f is easily seen to be continuous, we see that each horizontal line $X \times \{y\}$ is connected. A similar argument shows that each vertical line $\{x\} \times Y$ is connected as well. These two sets have a point in common, so their cross-shaped union

$$C_{xy} = X \times \{y\} \cup \{x\} \times Y$$

is connected itself. On the other hand, one has $X \times Y = \bigcup_x C_{xy}$ for any fixed $y \in Y$, so the product $X \times Y$ is connected as well. ■

Theorem 2.17 – Connected components are closed

Let (X, T) be a topological space. Then X is the disjoint union of its connected components and each connected component is closed in X.

Proof. Let C_x be the connected component of x for each $x \in X$. According to the previous theorem, $\overline{C}_x$ is also connected. Since C_x is the largest connected set that contains x, this implies $\overline{C}_x \subset C_x$, so the two sets are equal and C_x is closed.

Now, it is clear that X is the union of all connected components, as each element x is contained in C_x. Thus, it remains to show that the connected components are disjoint. Suppose that C_x, C_y have a point in common. Then $C_x \cup C_y$ is connected and it contains each of x, y. This actually implies that

$$C_x \cup C_y \subset C_x \quad \text{and} \quad C_x \cup C_y \subset C_y.$$

In particular, the connected components C_x, C_y must be equal. ■

Theorem 2.18 – Compactness and convergence

Suppose that X is a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let $\{x_n\}$ be a sequence in X and let $x \in X$ be arbitrary.

If the open ball $B(x, 1/k)$ contains infinitely many terms of the sequence for each $k > 0$, one may choose a term $x_{n_k} \in B(x, 1/k)$ for each $k > 0$ to obtain a subsequence that converges to x.

Assume this is not the case. Given any $x \in X$, we can then find some open ball $B(x, 1/k)$ that contains finitely many terms of the sequence. These open balls form an open cover of X and a finite subcover exists by compactness. Since the finite subcover contains only finitely many terms, the whole sequence contains finitely many terms. In particular, one of the terms appears infinitely many times and this gives rise to a constant, convergent subsequence. ■

Theorem 2.19 – Main facts about compact spaces

1. A compact subset of a Hausdorff space is closed.

Proof. Suppose X is Hausdorff and $A \subset X$ is compact. To show that $X - A$ is open, let $x \in X - A$ be given. Then for each $y \in A$ there exist disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. Since the sets V_y form an open cover of A, finitely many of them cover A by compactness. Suppose that $V_{y_1}, \ldots, V_{y_n}$ do and let

$$U = U_{y_1} \cap \cdots \cap U_{y_n}.$$

Since U does not intersect any V_{y_i}, it does not intersect A, either.

This shows that each $x \in X - A$ has a neighbourhood U which lies entirely within $X - A$. In other words, every element of $X - A$ lies in the interior of $X - A$. It easily follows that $X - A$ is equal to its interior, so this set is open and A is closed. ∎

Theorem 2.19 – Main facts about compact spaces

2. A closed subset of a compact space is compact.

Proof. Suppose X is compact and let $A \subset X$ be closed. To show that A is compact, suppose the sets U_i form an open cover of A. Adjoining $X - A$ to these sets gives an open cover of X. This has a finite subcover by compactness, so X is covered by finitely many of the sets U_i along with $X - A$. In particular, A itself is covered by finitely many of the sets U_i and so A is compact. ∎

Theorem 2.19 – Main facts about compact spaces

3. The interval $[a, b]$ is compact for all real numbers $a < b$.

Proof. Suppose $I_0 = [a, b]$ is not compact. Then some open sets U_i form an open cover of I_0 with no finite subcover. Divide I_0 into two closed intervals of equal length. At least one of them is not covered by finitely many U_i. Denote it by I_1 and proceed in this manner to get a sequence of closed intervals I_n that are not covered by finitely many U_i, while I_n has length $(b - a)/2^n$ and $I_n \supset I_{n+1}$ for all n.

The numbers $x_n = \min I_n$ form an increasing sequence which is also bounded. Let x denote its limit. We note that $x \in I_n$ for all n and $x \in U_j$ for some j. Since U_j is open, there exists some $\varepsilon > 0$ such that $I = (x - \varepsilon, x + \varepsilon)$ lies within U_j. Pick an integer n such that $(b - a)/2^n < \varepsilon$. The intervals I_n, I both contain x, while the length of I_n is less than ε. This implies that $I_n \subset I \subset U_j$, which is contrary to the fact that I_n is not covered by a single U_j. ∎

Theorem 2.19 – Main facts about compact spaces

4. The continuous image of a compact space is compact: if X is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.

Proof. Suppose the sets U_i form an open cover of $f(X)$. Since f is continuous, the inverse images $f^{-1}(U_i)$ must then form an open cover of X. In particular, finitely many of them cover X, say

$$X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n).$$

It easily follows that

$$f(X) = U_1 \cup \cdots \cup U_n.$$

Thus, finitely many U_i cover $f(X)$ and so $f(X)$ is compact. ∎

Theorem 2.19 – Main facts about compact spaces

5. If X is compact and $f: X \to \mathbb{R}$ is continuous, then f is bounded.

Proof. According to part 4, the image $f(X)$ is compact. Consider the open intervals $(-n, n)$ with $n \in \mathbb{N}$. These form an open cover of $f(X)$, so finitely many of them cover $f(X)$ and

$$f(X) \subset (-n_1, n_1) \cup \cdots \cup (-n_k, n_k)$$

for some positive integers $n_1, \ldots, n_k$. Letting N denote the largest of these integers, we conclude that

$$f(X) \subset (-N, N).$$

In other words, $|f(x)| < N$ for all $x \in X$ and so f is bounded. ∎

Theorem 2.19 – Main facts about compact spaces

6. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then there exist points $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.

Proof. We note that f is bounded by part 5. Let m and M denote its infimum and supremum, respectively. Then $m \leq f(x) \leq M$ for all $x \in X$ and we need to show that neither inequality is strict. We only prove this for the first inequality, as the other one is similar.

Suppose that $f(x) > m$ for all $x \in X$. Then the function

$$g(x) = \frac{1}{f(x) - m}$$

is positive and continuous on X, so it must be bounded. Let $R > 0$ be a real number such that $g(x) \leq R$ for all $x \in X$. Then it easily follows that $f(x) \geq m + 1/R$ for all $x \in X$. This makes $m + 1/R$ a lower bound of f, contrary to the fact that $m = \inf f$. ∎

Theorem 2.19 – Main facts about compact spaces

7 The product of two compact spaces is compact.

Proof. Suppose that X, Y are compact and consider an open cover of $X \times Y$. We may assume it consists of the sets $W_i = U_i \times V_i$, where each U_i is open in X and each V_i is open in Y.

Given any $y \in Y$, one may define a surjective function

$$f \colon X \to X \times \{y\}, \qquad f(x) = (x, y).$$

Then f is continuous, so $X \times \{y\}$ is compact and thus covered by finitely many sets $W_i = U_i \times V_i$. Let $V(y)$ denote the intersection of the corresponding sets V_i. Then $V(y)$ is a neighbourhood of y such that $X \times V(y)$ is covered by finitely many sets $W_i = U_i \times V_i$.

The open sets $V(y)$ obtained above form an open cover of Y, so finitely many of them cover Y. Since each $X \times V(y)$ is covered by finitely many sets W_i, the same is true for the product $X \times Y$. ■

Theorem 2.20 – Heine-Borel theorem

A subset of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Proof. Suppose that $A \subset \mathbb{R}^k$ is compact. Then A is a compact subset of a Hausdorff space, hence also closed. Since $\mathbb{R}^k$ is covered by the open boxes $(-n, n)^k$ with $n \in \mathbb{N}$, finitely many of these boxes must cover A, so A is bounded as well.

Conversely, suppose that $A \subset \mathbb{R}^k$ is closed and bounded. Then there is a positive integer N such that A is contained in the closed box $[-N, N]^k$. We note that this box is compact because a finite product of compact spaces is compact by induction. In particular, A is a closed subset of a compact space, hence also compact. ■

Theorem 2.21 – Main facts about homeomorphisms

1. Consider two homeomorphic topological spaces. If one of them is connected or compact or Hausdorff, then so is the other.

Proof. Suppose that $f\colon X \to Y$ is a homeomorphism. Then f and its inverse f^{-1} are both continuous. Since the continuous image of a connected space is connected, Y is connected if and only if X is connected. The same argument applies for compact spaces because the continuous image of a compact space is compact.

Finally, suppose that Y is Hausdorff and $x_1, x_2 \in X$ are distinct. Since f is bijective, the images $f(x_1), f(x_2) \in Y$ are also distinct, so they are contained in disjoint neighbourhoods U, V. The inverse images of those are then disjoint neighbourhoods of x_1, x_2.

This shows that X is Hausdorff, if Y is Hausdorff. The converse follows by applying this result to the inverse function f^{-1}. ∎

Theorem 2.21 – Main facts about homeomorphisms

2. Suppose $f\colon X \to Y$ is bijective and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We need only show that f^{-1} is continuous. Suppose U is open in X. Then $X - U$ is closed in the compact space X, so it is compact. Since f is continuous, it follows that $f(X-U)$ is a compact subset of the Hausdorff space Y, so it is closed in Y.

On the other hand, the fact that f is bijective implies that

$$\begin{aligned} y \in f(X-U) \quad &\Longleftrightarrow \quad y = f(x) \quad \text{for some } x \notin U \\ &\Longleftrightarrow \quad y \notin f(U). \end{aligned}$$

In other words, one has $f(X-U) = Y - f(U)$. Since this set is closed in Y, we conclude that $f(U)$ is open in Y, as needed. ■

Theorem 2.22 – Main facts about uniform continuity

1. Every Lipschitz continuous function is uniformly continuous.

Proof. Suppose that $f\colon X \to Y$ is a Lipschitz continuous function. Then there exists a constant $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) \quad \text{for all } x, y \in X.$$

Let $\varepsilon > 0$ be given. When $L > 0$, we can take $\delta = \varepsilon / L$ to find that

$$\begin{aligned} d_X(x, y) < \delta \quad &\Longrightarrow \quad d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) \\ &\Longrightarrow \quad d_Y(f(x), f(y)) < L \cdot \delta = \varepsilon. \end{aligned}$$

This shows that f is uniformly continuous on X. When $L = 0$, one still has $d_Y(f(x), f(y)) \leq 0 < \varepsilon$, so the same conclusion holds. ∎

Theorem 2.22 – Main facts about uniform continuity

2. Every uniformly continuous function is continuous.

Proof. To say that $f\colon X \to Y$ is uniformly continuous on X is to say that given any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$d_X(x,y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon$$

for all $x, y \in X$. To say that f is continuous at a point $x \in X$ is to say that the same condition holds for all $y \in X$. Thus, uniform continuity on X trivially implies continuity at all points of X. ∎

Theorem 2.22 – Main facts about uniform continuity

3. When X is compact, a function $f\colon X \to Y$ is continuous on X if and only if it is uniformly continuous on X.

Proof, part 1. Uniform continuity on X trivially implies continuity at all points of X. To prove the converse, let $\varepsilon > 0$ be given. Then for each point $x \in X$ there exists some $\delta(x) > 0$ such that

$$d_X(x, y) < \delta(x) \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon/2.$$

Since the open balls $B(x, \delta(x)/2)$ form an open cover of X, finitely many of them cover X by compactness, say

$$X = \bigcup_{i=1}^{n} B(x_i, \delta(x_i)/2).$$

Letting $\delta > 0$ be the smallest of the finitely many numbers $\delta(x_i)/2$, we shall now show that f is uniformly continuous on X.

Theorem 2.22 – Main facts about uniform continuity

3. When X is compact, a function $f\colon X \to Y$ is continuous on X if and only if it is uniformly continuous on X.

Proof, part 2. Suppose $x, y \in X$ are such that $d_X(x, y) < \delta$ and note that $x \in B(x_i, \delta(x_i)/2)$ for some $1 \leq i \leq n$. We thus have

$$d_X(y, x_i) \leq d_X(y, x) + d_X(x, x_i) < \delta + \delta(x_i)/2 \leq \delta(x_i).$$

Once we now recall the definition of $\delta(x_i)$, we may conclude that

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) < \varepsilon.$$

This shows that the function f is uniformly continuous on X. ■