Chapter 1. Metric spaces

Proofs covered in class

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Theorem 1.1 – Technical inequalities

Suppose that $x, y \geq 0$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be arbitrary vectors in $\mathbb{R}^k$.

1. **Young's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Proof. Let $y \geq 0$ be fixed and consider the function

$$f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy, \qquad x \geq 0.$$

Since $f'(x) = x^{p-1} - y$, this function is decreasing when $x^{p-1} < y$ and increasing when $x^{p-1} > y$, so it attains its minimum value at the point $x_* = y^{1/(p-1)}$. One may now easily check that $f(x_*) = 0$ and this implies that $f(x) \geq 0$ for all $x \geq 0$, as needed. ∎

Theorem 1.1 – Technical inequalities

Suppose that $x, y \geq 0$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be arbitrary vectors in $\mathbb{R}^k$.

2. **Hölder's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{k} |a_i| \cdot |b_i| \leq \left[\sum_{i=1}^{k} |a_i|^p \right]^{1/p} \left[\sum_{i=1}^{k} |b_i|^q \right]^{1/q}.$$

Proof. The result is clear when either $\boldsymbol{a}$ or $\boldsymbol{b}$ is zero. When $\boldsymbol{a}, \boldsymbol{b}$ are both nonzero, we need to show that $\sum_{i=1}^{k} x_i y_i \leq 1$, where

$$x_i = |a_i| \cdot \left[\sum_{i=1}^{k} |a_i|^p \right]^{-1/p} \quad \text{and} \quad y_i = |b_i| \cdot \left[\sum_{i=1}^{k} |b_i|^q \right]^{-1/q}.$$

Young's inequality gives $x_i y_i \leq \frac{1}{p} x_i^p + \frac{1}{q} y_i^q$ for each i. Once we now add these inequalities, we find that $\sum_{i=1}^{k} x_i y_i \leq \frac{1}{p} + \frac{1}{q} = 1$. ■

Theorem 1.1 – Technical inequalities

Suppose that $x, y \geq 0$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be arbitrary vectors in $\mathbb{R}^k$.

3. **Minkowski's inequality**: If $p > 1$, then

$$d_p(\boldsymbol{a}, \boldsymbol{b}) \leq d_p(\boldsymbol{a}, \boldsymbol{c}) + d_p(\boldsymbol{c}, \boldsymbol{b}).$$

Proof. First, we use the triangle inequality in $\mathbb{R}$ to find that

$$d_p(\boldsymbol{a}, \boldsymbol{b})^p \leq \sum_{i=1}^{k} |a_i - c_i| \, |a_i - b_i|^{p-1} + \sum_{i=1}^{k} |c_i - b_i| \, |a_i - b_i|^{p-1}.$$

Letting $q = \frac{p}{p-1}$, we also have $\frac{1}{p} + \frac{1}{q} = 1$, so Hölder's inequality gives

$$d_p(\boldsymbol{a}, \boldsymbol{b})^p \leq d_p(\boldsymbol{a}, \boldsymbol{c}) \, d_p(\boldsymbol{a}, \boldsymbol{b})^{p-1} + d_p(\boldsymbol{c}, \boldsymbol{b}) \, d_p(\boldsymbol{a}, \boldsymbol{b})^{p-1}.$$

This already implies Minkowski's inequality whenever $d_p(\boldsymbol{a}, \boldsymbol{b}) \neq 0$ and the inequality holds trivially whenever $d_p(\boldsymbol{a}, \boldsymbol{b}) = 0$. ∎

Theorem 1.2 – Main facts about open sets

1. If X is a metric space, then both $\varnothing$ and X are open in X.
2. Arbitrary unions of open sets are open.

Proof. First, we prove 1. The definition of an open set is satisfied by every point in the empty set simply because there is no point in the empty set. This means that $\varnothing$ is open in X. To show that X is open in X, let $x \in X$ and consider the open ball $B(x, 1)$. This is a subset of X by definition, so X is open in X.

Next, we prove 2. Suppose that the sets U_i are open in X and let x be a point in their union. Then $x \in U_i$ for some i. Since U_i is open in X, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_i$. This implies that $B(x, \varepsilon) \subset \bigcup_i U_i$, so the union is open in X as well. ■

Theorem 1.2 – Main facts about open sets

3. Finite intersections of open sets are open.

Proof. Suppose that the sets U_i are open in X and let x be a point in their intersection. Then $x \in U_i$ for all i. Since U_i is open in X for each i, there exists $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subset U_i$. Let ε be the smallest of the finitely many numbers ε_i. Then $\varepsilon > 0$ and we have

$$B(x, \varepsilon) \subset B(x, \varepsilon_i) \subset U_i$$

for all i. This shows that the intersection is open in X as well. ■

Theorem 1.2 – Main facts about open sets

4. Every open ball is an open set.

Proof. Consider the ball $B(x, \varepsilon)$ and let $y \in B(x, \varepsilon)$ be arbitrary. Then $d(x, y) < \varepsilon$ and so the number $r = \varepsilon - d(x, y)$ is positive. To finish the proof, it suffices to show that $B(y, r) \subset B(x, \varepsilon)$.

Suppose then that $z \in B(y, r)$. Since $d(y, z) < r$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r = \varepsilon$$

and so $z \in B(x, \varepsilon)$. This shows that $B(y, r) \subset B(x, \varepsilon)$. ■

Theorem 1.2 – Main facts about open sets

5. A set is open if and only if it is a union of open balls.

Proof. Suppose first that U is a union of open balls. Then U is a union of open sets by part 4, so it is open itself by part 2.

Conversely, suppose that U is an open set. Given any $x \in U$, we can then find some $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$. This gives

$$\{x\} \subset B(x, \varepsilon_x) \subset U$$

and we can take the union over all possible $x \in U$ to find that

$$U \subset \bigcup_{x \in U} B(x, \varepsilon_x) \subset U.$$

Thus, U is a union of open balls and the proof is complete. ∎

Theorem 1.3 – Limits are unique

The limit of a sequence in a metric space is unique. In other words, no sequence may converge to two different limits.

Proof. Suppose $\{x_n\}$ is a convergent sequence which converges to two different limits $x \neq y$. Then $\varepsilon = \frac{1}{2}d(x, y)$ is positive, so there exist integers N_1, N_2 such that

$$d(x_n, x) < \varepsilon \quad \text{for all } n \geq N_1,$$
$$d(x_n, y) < \varepsilon \quad \text{for all } n \geq N_2.$$

Setting $N = \max\{N_1, N_2\}$ for convenience, we conclude that

$$2\varepsilon = d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\varepsilon$$

for all $n \geq N$. This is a contradiction, so the limit is unique. ■

Theorem 1.4 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.

Proof. Let $\{x_n\}$ be a sequence of points of A that converges and let x be its limit. Suppose $x \in X - A$. Since $X - A$ is open, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X - A$. Since x_n converges to x, there also exists an integer N such that

$$x_n \in B(x, \varepsilon) \quad \text{for all } n \geq N.$$

This implies $x_n \in X - A$ for all $n \geq N$, which is contrary to the fact that $x_n \in A$ for all $n \in \mathbb{N}$. In particular, we must have $x \in A$. ■

Theorem 1.4 – Main facts about closed sets

2. Both $\varnothing$ and X are closed in X.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

Proof. First, we prove 2. Since the sets $\varnothing, X$ are both open in X, their complements $X, \varnothing$ are both closed in X.

To prove 3 and 4, one needs to use De Morgan's laws

$$X - \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (X - U_i), \qquad X - \bigcap_{i} U_i = \bigcup_{i} (X - U_i).$$

For instance, consider finitely many sets U_i which are closed in X. Their complements $X - U_i$ are then open in X, so the same is true for their intersection. Using the first De Morgan's law, we conclude that the union of the sets U_i is closed in X. This proves 3 and one may similarly prove 4 using the second De Morgan's law. ∎

Theorem 1.5 – Composition of continuous functions

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between metric spaces. Then the composition $g \circ f\colon X \to Z$ is continuous.

Proof. We show that $g \circ f$ is continuous at any $x \in X$. Let $\varepsilon > 0$ be given. Since g is continuous at $f(x)$, there exists $\delta > 0$ with

$$d_Y(f(x), y) < \delta \quad \Longrightarrow \quad d_Z(g(f(x)), g(y)) < \varepsilon.$$

Since f is continuous at x, there also exists $\delta' > 0$ with

$$d_X(x, x') < \delta' \quad \Longrightarrow \quad d_Y(f(x), f(x')) < \delta.$$

Once we now combine the last two equations, we find that

$$d_X(x, x') < \delta' \quad \Longrightarrow \quad d_Z(g(f(x)), g(f(x'))) < \varepsilon.$$

This shows that the composition $g \circ f$ is continuous at x. ■

Theorem 1.6 – Continuity and sequences

Suppose $f\colon X \to Y$ is a continuous function between metric spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Proof. Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon.$$

Since x_n converges to x, there also exists an integer N such that

$$d_X(x_n, x) < \delta \quad \text{for all } n \geq N.$$

Once we now combine the last two equations, we find that

$$d_Y(f(x_n), f(x)) < \varepsilon \quad \text{for all } n \geq N.$$

This shows that $f(x_n)$ converges to $f(x)$, as needed. ■

Theorem 1.7 – Continuity and open sets

A function $f\colon X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Proof. First, suppose f is continuous and let U be open in Y. To show that $f^{-1}(U)$ is open, let $x \in f^{-1}(U)$. Then $f(x) \in U$ and so there exists $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset U$. By continuity, there also exists $\delta > 0$ such that

$$y \in B(x, \delta) \quad \Longrightarrow \quad f(y) \in B(f(x), \varepsilon). \tag{$*$}$$

This implies that $B(x, \delta) \subset f^{-1}(U)$ and so $f^{-1}(U)$ is open.

Conversely, suppose $f^{-1}(U)$ is open in X for each set U which is open in Y. Let $x \in X$ and $\varepsilon > 0$ be given. Setting $U = B(f(x), \varepsilon)$, we find that $f^{-1}(U)$ is open in X. This gives $B(x, \delta) \subset f^{-1}(U)$ for some $\delta > 0$, so the definition $(*)$ of continuity holds. ∎

Theorem 1.8 – Main facts about Lipschitz continuity

1. Every Lipschitz continuous function is continuous.

Proof. Suppose $f\colon X \to Y$ is a Lipschitz continuous function. To show that f is continuous at all points $x \in X$, let $\varepsilon > 0$ be given. Since f is Lipschitz continuous, we have

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$$

for some $L \geq 0$. When $L > 0$, we can take $\delta = \varepsilon / L$ to find that

$$\begin{aligned} d_X(x,y) < \delta \quad &\Longrightarrow \quad d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) \\ &\Longrightarrow \quad d_Y(f(x), f(y)) < L \cdot \delta = \varepsilon. \end{aligned}$$

When $L = 0$, one always has $d_Y(f(x), f(y)) \leq 0 < \varepsilon$, so the choice of δ is irrelevant. Thus, f is continuous at x in any case. ∎

Theorem 1.8 – Main facts about Lipschitz continuity

2. If a function $f\colon [a,b] \to \mathbb{R}$ is differentiable and its derivative is bounded, then f is Lipschitz continuous on $[a,b]$.

Proof. Suppose $|f'(x)| \leq M$ for all $x \in [a,b]$ and let $x, y \in [a,b]$ be arbitrary. Using the mean value theorem, we can then write

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y|$$

for some c between x and y. This obviously gives

$$|f(x) - f(y)| \leq M \cdot |x - y|$$

for all $x, y \in [a,b]$ and so f is Lipschitz continuous on $[a,b]$. ■

Theorem 1.9 – Pointwise and uniform convergence

1. To say that $f_n(x) \to f(x)$ pointwise is to say that

$$|f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Proof. By definition, to say that $f_n(x) \to f(x)$ pointwise is to say that, given any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N.$$

This is the case if and only if $|f_n(x) - f(x)| \to 0$ as $n \to \infty$. ∎

Theorem 1.9 – Pointwise and uniform convergence

2. To say that $f_n \to f$ uniformly on X is to say that

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Proof. Let $M_n = \sup_{x \in X} |f_n(x) - f(x)|$ for convenience. Suppose that $M_n \to 0$ as $n \to \infty$ and let $\varepsilon > 0$ be given. Then there is an integer N such that $M_n < \varepsilon$ for all $n \geq N$, so we also have

$$|f_n(x) - f(x)| \leq M_n < \varepsilon$$

for all $n \geq N$ and all $x \in X$. In particular, $f_n \to f$ uniformly on X.

Conversely, suppose $f_n \to f$ uniformly on X and let $\varepsilon > 0$ be given. Then there is an integer N such that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and all $x \in X$. Taking the supremum over all $x \in X$, we get $M_n \leq \varepsilon/2 < \varepsilon$ for all $n \geq N$, so $M_n \to 0$ as $n \to \infty$. ■

Theorem 1.10 – Uniform limit of continuous functions

The uniform limit of continuous functions is continuous: if each f_n is continuous and $f_n \to f$ uniformly on X, then f is continuous on X.

Proof. Let $\varepsilon > 0$ be given. Then there is an integer N such that

$$|f_n(x) - f(x)| < \varepsilon/3 \quad \text{for all } n \geq N \text{ and all } x \in X.$$

Since f_N is continuous at the point x, there exists $\delta > 0$ such that

$$d(x,y) < \delta \quad \Longrightarrow \quad |f_N(x) - f_N(y)| < \varepsilon/3.$$

Suppose now that $d(x,y) < \delta$. Then the triangle inequality gives

$$\begin{aligned} |f(x) - f(y)| \leq & \, |f(x) - f_N(x)| \\ & + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \varepsilon. \end{aligned}$$

In particular, f is continuous at each point $x \in X$, as needed. ■

Theorem 1.11 – Convergent implies Cauchy

In a metric space, every convergent sequence is a Cauchy sequence.

Proof. Suppose that $\{x_n\}$ is a sequence which converges to x and let $\varepsilon > 0$ be given. Then there exists an integer N such that

$$d(x_n, x) < \varepsilon/2 \quad \text{for all } n \geq N.$$

Using this fact and the triangle inequality, we conclude that

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon$$

for all $m, n \geq N$. This shows that the sequence is Cauchy. ∎

Theorem 1.12 – Cauchy implies bounded

In a metric space, every Cauchy sequence is bounded.

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space X. Then there exists an integer N such that

$$d(x_m, x_n) < 1 \quad \text{for all } m, n \geq N.$$

Taking $n = N$, in particular, we find that

$$d(x_m, x_N) < 1 \quad \text{for all } m \geq N.$$

Thus, every term starting with the Nth term lies in the open ball of radius 1 around x_N. As for the other terms, we have

$$R = \max_{1 \leq m < N} d(x_m, x_N) \quad \Longrightarrow \quad d(x_m, x_N) \leq R$$

for all $m < N$. This implies that $d(x_i, x_N) < R + 1$ for all i. ■

Theorem 1.13 – Cauchy sequence with convergent subsequence

Suppose (X, d) is a metric space and let $\{x_n\}$ be a Cauchy sequence in X that has a convergent subsequence. Then $\{x_n\}$ converges itself.

Proof. Suppose that $\{x_n\}$ is Cauchy and $\{x_{n_k}\}$ converges to x. We claim that $\{x_n\}$ converges to x as well. Let $\varepsilon > 0$ be given. Then there exist integers N_1, N_2 such that

$$d(x_m, x_n) < \varepsilon/2 \quad \text{for all } m, n \geq N_1,$$
$$d(x_{n_k}, x) < \varepsilon/2 \quad \text{for all } n_k \geq N_2.$$

Set $N = \max\{N_1, N_2\}$ and fix some $n_k \geq N$. Then we have

$$d(x_m, x) \leq d(x_m, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$$

for all $m \geq N$ and this implies that $x_m \to x$ as $m \to \infty$. ■

Theorem 1.14 – Completeness of $\mathbb{R}$

1. Every sequence in $\mathbb{R}$ which is monotonic and bounded converges.

Proof. Suppose that $\{x_n\}$ is increasing and bounded, as the case that $\{x_n\}$ is decreasing is similar. We claim that $\alpha = \sup x_n$ is the limit of the sequence. Let $\varepsilon > 0$ be given. Since $\alpha - \varepsilon$ is less than the least upper bound, there exists a term x_N such that

$$\alpha - \varepsilon < x_N \leq \alpha.$$

Since the sequence $\{x_n\}$ is increasing, this actually gives

$$\alpha - \varepsilon < x_n \leq \alpha$$

for all $n \geq N$. In particular, $|x_n - \alpha| = \alpha - x_n < \varepsilon$ for all $n \geq N$ and this implies that $x_n \to \alpha$ as $n \to \infty$. ∎

Theorem 1.14 – Completeness of $\mathbb{R}$

2. **Bolzano-Weierstrass theorem**: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Suppose $\{x_n\}$ is a bounded sequence. We say that x_k is a peak point, if all subsequent terms are smaller than x_k. Let us now consider two cases. If there are infinitely many peak points, then there is a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ consisting of peak points and this is decreasing, hence also convergent.

Otherwise, there are finitely many peak points. Choose N_1 large enough so that x_{N_1} is not a peak point. Then there is a subsequent term x_{N_2} which is larger. Since x_{N_2} is not a peak point, there is a further term which is even larger. Proceeding in this way, we obtain a subsequence which is increasing, hence also convergent. ■

Theorem 1.14 – Completeness of $\mathbb{R}$

③ The set $\mathbb{R}$ of all real numbers is a complete metric space.

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. Then it is also bounded, so it has a convergent subsequence by part ②. Using Theorem 1.13, we conclude that $\{x_n\}$ converges itself. This shows that every Cauchy sequence in $\mathbb{R}$ converges, so $\mathbb{R}$ is complete. ■

Theorem 1.15 – Examples of complete metric spaces

1. The space $\mathbb{R}^k$ is complete with respect to its usual metric.

Proof. Let $\boldsymbol{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nk})$ be a Cauchy sequence in $\mathbb{R}^k$ and let $\varepsilon > 0$ be given. Each x_{ni} forms a Cauchy sequence in $\mathbb{R}$, as

$$|x_{mi} - x_{ni}|^2 \leq \sum_{i=1}^{k} |x_{mi} - x_{ni}|^2 = d_2(\boldsymbol{x}_m, \boldsymbol{x}_n)^2.$$

Let x_i be the limit of x_{ni} as $n \to \infty$. Then there is an integer N such that $|x_{ni} - x_i| < \varepsilon/\sqrt{k}$ for all $n \geq N$ and each $1 \leq i \leq k$. Once we now set $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$, we find that

$$d_2(\boldsymbol{x}_n, \boldsymbol{x})^2 = \sum_{i=1}^{k} |x_{ni} - x_i|^2 < \sum_{i=1}^{k} \varepsilon^2/k = \varepsilon^2$$

for all $n \geq N$. Thus, $\boldsymbol{x}_n$ is convergent and so $\mathbb{R}^k$ is complete. ■

Theorem 1.15 – Examples of complete metric spaces

2. The space $C[a,b]$ is complete with respect to the d_∞ metric.

Proof. Suppose that $\{f_n\}$ is a Cauchy sequence in $C[a,b]$. Given any $\varepsilon > 0$, we can then find an integer N such that

$$|f_m(x) - f_n(x)| < \varepsilon/2 \quad \text{for all } m, n \geq N \text{ and all } x \in [a,b].$$

Thus, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in [a,b]$. Denote the limit of this sequence by $f(x)$. Letting $m \to \infty$, we get

$$|f(x) - f_n(x)| \leq \varepsilon/2 \quad \text{for all } n \geq N \text{ and all } x \in [a,b].$$

Taking the supremum over all $x \in [a,b]$ now gives $d_\infty(f, f_n) < \varepsilon$ for all $n \geq N$. This shows that $d_\infty(f, f_n) \to 0$ as $n \to \infty$ which also means that $f_n \to f$ uniformly on $[a,b]$. Since f is the uniform limit of continuous functions, f is continuous as well. ■

Theorem 1.16 – Subsets of a complete metric space

Suppose (X, d) is a complete metric space and let $A \subset X$. Then A is complete if and only if A is closed in X.

Proof. First, suppose $A \subset X$ is closed and let $\{x_n\}$ be a Cauchy sequence in A. This sequence converges in X by completeness. In view of Theorem 1.4, the limit of the sequence lies in A. That is, the Cauchy sequence converges in A and so A is complete.

Conversely, suppose $A \subset X$ is complete. To show that $X - A$ is open, let $x \in X - A$ and consider the open balls $B(x, 1/n)$ for each $n \in \mathbb{N}$. If one of those lies entirely in $X - A$, then $X - A$ is open and the proof is complete. Otherwise, each $B(x, 1/n)$ must contain a point $x_n \in A$. Noting that $d(x_n, x) < 1/n$ for each n, we see that $\{x_n\}$ converges to $x \in X - A$. This contradicts our initial assumption that the subset A is complete. ■

Theorem 1.17 – Banach's fixed point theorem

If $f\colon X \to X$ is a contraction on a complete metric space X, then f has a unique fixed point, namely a unique point x with $f(x) = x$.

Proof, part 1. First, we prove uniqueness. Suppose that $x \neq y$ are both fixed points. Since f is a contraction, we then have

$$d(x, y) = d(f(x), f(y)) \leq \alpha \cdot d(x, y)$$

for some $0 \leq \alpha < 1$. This leads to the contradiction $\alpha \geq 1$.

It remains to show existence. Let $x \in X$ be arbitrary and define a sequence by setting $x_1 = x$ and $x_{n+1} = f(x_n)$ for each $n \geq 1$. If this sequence is actually Cauchy, then it converges by completeness and its limit y is a fixed point of f because

$$y = \lim_{n\to\infty} x_n \quad \Longrightarrow \quad f(y) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = y.$$

Theorem 1.17 – Banach's fixed point theorem

If $f\colon X \to X$ is a contraction on a complete metric space X, then f has a unique fixed point, namely a unique point x with $f(x) = x$.

Proof, part 2. It remains to show that our sequence is Cauchy. Since f is a contraction and $x_{n+1} = f(x_n)$ for each n, we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &= \sum_{i=n}^{n+k-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+k-1} \alpha^{i-1} \cdot d(x_1, x_2) \\ &\leq \sum_{i=n}^{\infty} \alpha^{i-1} \cdot d(x_1, x_2) = \frac{\alpha^{n-1}}{1-\alpha} \cdot d(x_1, x_2). \end{aligned}$$

The right hand side goes to zero as $n \to \infty$, so the same is true for the left hand side. This shows that our sequence is Cauchy. ∎

Theorem 1.18 – Existence and uniqueness of solutions

Consider an initial value problem of the form

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

If f is continuous in t and Lipschitz continuous in y, then there exists a unique solution $y(t)$ which is defined on $[0, \varepsilon]$ for some $\varepsilon > 0$.

Proof, part 1. It is easy to see that $y(t)$ is a solution if and only if

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds.$$

Let us denote the right hand side by $\mathcal{A}(y(t))$. Then every solution corresponds to a fixed point of $\mathcal{A}$, so it suffices to show that $\mathcal{A}$ is a contraction on a complete metric space X. In fact, $X = C[0, \varepsilon]$ is complete with respect to the d_∞ metric for any $\varepsilon > 0$.

Theorem 1.18 – Existence and uniqueness of solutions

Consider an initial value problem of the form

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

If f is continuous in t and Lipschitz continuous in y, then there exists a unique solution $y(t)$ which is defined on $[0, \varepsilon]$ for some $\varepsilon > 0$.

Proof, part 2. To show that $\mathcal{A}$ is a contraction, we note that

$$\left| f(s, y(s)) - f(s, z(s)) \right| \leq L\, |y(s) - z(s)| \leq L\, d_\infty(y, z).$$

Fix some $0 < \varepsilon < 1/L$ and let $y(t), z(t) \in C[0, \varepsilon]$. We then have

$$\left| \mathcal{A}(y(t)) - \mathcal{A}(z(t)) \right| \leq L\, d_\infty(y, z) \int_0^t ds \leq \varepsilon L\, d_\infty(y, z)$$

for all $t \in [0, \varepsilon]$ and this implies that $\mathcal{A}$ is a contraction, indeed. ■

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f\colon X \to X'$ such that X' is complete.

Proof, part 1. Suppose $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X. Using the triangle inequality, one easily finds that

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_m, y_m) &\leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m). \end{aligned}$$

Rearranging terms, we now combine these equations to get

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n).$$

This implies that the sequence of real numbers $\{d(x_n, y_n)\}$ is also Cauchy, so this sequence must converge by completeness. We now use this fact to define the completion X' of the metric space X.

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f\colon X \to X'$ such that X' is complete.

Proof, part 2. Consider the set of all Cauchy sequences in X. One may define a relation on this set by setting

$$\{x_n\} \sim \{y_n\} \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

This relation is clearly reflexive and symmetric. To see that it is also transitive, suppose that $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Since

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n),$$

it easily follows that $\{x_n\} \sim \{z_n\}$ as well. This means that $\sim$ is an equivalence relation on the set of Cauchy sequences in X. Let us denote by X' the set of all equivalence classes. We now define a metric d' on X' and a distance preserving map $f\colon X \to X'$.

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Proof, part 3. One may define a metric on X' by letting

$$d'([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n).$$

Suppose that $\{x_n\} \sim \{a_n\}$ and $\{y_n\} \sim \{b_n\}$. Using the inequalities

$$d(a_n, b_n) \leq d(a_n, x_n) + d(x_n, y_n) + d(y_n, b_n),$$
$$d(x_n, y_n) \leq d(x_n, a_n) + d(a_n, b_n) + d(b_n, y_n),$$

we see that $d(a_n, b_n)$ and $d(x_n, y_n)$ have the same limit as $n \to \infty$. Thus, d' is well-defined. Since each $x \in X$ gives rise to a constant sequence, one may also define $f\colon X \to X'$ by $f(x) = [x]$. Then f is distance preserving, so it remains to show that X' is complete.

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f\colon X \to X'$ such that X' is complete.

Proof, part 4. A Cauchy sequence in X' has the form $\{[x_n^i]\}_{i=1}^{\infty}$, where $\{x_n^i\}$ is a Cauchy sequence in X for each i. In particular, we can find an integer N_i for each i such that

$$d(x_m^i, x_n^i) < 1/i \quad \text{for all } m, n \geq N_i.$$

Consider the sequence $\{y_n\}$ defined by $y_n = x_{N_n}^n$. Then we have

$$\begin{aligned} d(y_m, y_n) &= d(x_{N_m}^m, x_{N_n}^n) \\ &\leq d(x_{N_m}^m, x_p^m) + d(x_p^m, x_p^n) + d(x_p^n, x_{N_n}^n) \end{aligned}$$

for all m, n, p and this implies that $\{y_n\}$ is Cauchy. Once we now show that $[x_n^i] \to [y_n]$ as $i \to \infty$, the completeness of X' will follow.

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f: X \to X'$ such that X' is complete.

Proof, part 5. We now show that $[x_n^i] \to [y_n]$ as $i \to \infty$. Let $\varepsilon > 0$ be given and recall that $\{y_n\}$ is a Cauchy sequence with $y_n = x_{N_n}^n$. In particular, there exists an integer N such that

$$d(y_i, y_n) < \varepsilon/3 \quad \text{for all } i, n \geq N.$$

When $i \geq \max\{N, 3/\varepsilon\}$ and $n \geq \max\{N_i, N\}$, we must then have

$$\begin{aligned} d(x_n^i, y_n) &\leq d(x_n^i, y_i) + d(y_i, y_n) \\ &= d(x_n^i, x_{N_i}^i) + d(y_i, y_n) \\ &< 1/i + \varepsilon/3 \leq 2\varepsilon/3. \end{aligned}$$

Thus, $[x_n^i] \to [y_n]$ as $i \to \infty$ and the proof is finally complete. ■