Chapter 3. Normed vector spaces

Lecture notes for MA2223

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The definition of a norm

Definition – Norm

Suppose X is a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A norm on X is a real-valued function $||\boldsymbol{x}||$ with the following properties.

1. Zero vector: $||\boldsymbol{x}|| = 0$ if and only if $\boldsymbol{x} = 0$.
2. Scalar factors: $||\lambda\boldsymbol{x}|| = |\lambda| \cdot ||\boldsymbol{x}||$ for all $\lambda \in \mathbb{F}$ and all $\boldsymbol{x} \in X$.
3. Triangle inequality: $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$.

- A normed vector space $(X, ||\cdot||)$ consists of a vector space X and a norm $||\boldsymbol{x}||$. One generally thinks of $||\boldsymbol{x}||$ as the length of $\boldsymbol{x}$.
- It is easy to check that every norm satisfies $||\boldsymbol{x}|| \geq 0$ for all $\boldsymbol{x} \in X$.
- Every normed vector space $(X, ||\cdot||)$ is also a metric space (X, d), as one may define a metric d using the formula $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||$. This particular metric is said to be induced by the norm.

Examples of normed vector spaces

- Given any $p \geq 1$, we can define a norm on $\mathbb{R}^k$ by letting

$$||\boldsymbol{x}||_p = \left[\sum_{i=1}^{k} |x_i|^p\right]^{1/p}.$$

- The space $C[a,b]$ has a similar norm for any $p \geq 1$, namely

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{1/p}.$$

- There is also a norm on $\mathbb{R}^k$ for the case $p = \infty$. It is defined by

$$||\boldsymbol{x}||_\infty = \max_{1 \leq i \leq k} |x_i|.$$

- Finally, there is a similar norm on $C[a,b]$ which is given by

$$||f||_\infty = \max_{a \leq x \leq b} |f(x)|.$$

Sequence spaces ℓ^p

- The space ℓ^p consists of all real sequences $\boldsymbol{x} = \{x_n\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

It is a normed vector space for any $p \geq 1$ and its norm is given by

$$||\boldsymbol{x}||_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}.$$

- The space ℓ^∞ consists of all bounded real sequences $\boldsymbol{x} = \{x_n\}$. It is a normed vector space and its norm is given by

$$||\boldsymbol{x}||_\infty = \sup_{n \geq 1} |x_n|.$$

- The space c_0 consists of all real sequences $\{x_n\}$ which converge to 0. It is easily seen to be a subspace of ℓ^∞.

Continuity of operations

Theorem 3.1 – Product norm

Suppose X, Y are normed vector spaces. Then one may define a norm on the product $X \times Y$ by letting $||(\boldsymbol{x}, \boldsymbol{y})|| = ||\boldsymbol{x}|| + ||\boldsymbol{y}||$.

Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space X.

1. The norm $f(\boldsymbol{x}) = ||\boldsymbol{x}||$, where $\boldsymbol{x} \in X$.
2. The vector addition $g(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in X$.
3. The scalar multiplication $h(\lambda, \boldsymbol{x}) = \lambda \boldsymbol{x}$, where $\lambda \in \mathbb{F}$ and $\boldsymbol{x} \in X$.

- We shall mainly use this theorem to justify computations such as

$$\lim_{n\to\infty} ||\boldsymbol{x}_n|| = \left|\left| \lim_{n\to\infty} \boldsymbol{x}_n \right|\right|.$$

Bounded linear operators

Definition – Bounded, linear, continuous

Let X, Y be normed vector spaces over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1. A function $T\colon X \to Y$ is called a linear operator, if

$$T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), \qquad T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x})$$

   for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all scalars $\lambda \in \mathbb{F}$.
2. A function $T\colon X \to Y$ is called bounded, if there exists a real number $M > 0$ such that $||T(\boldsymbol{x})|| \leq M||\boldsymbol{x}||$ for all $\boldsymbol{x} \in X$.
3. A function $T\colon X \to Y$ is called continuous, if it is continuous with respect to the metrics which are induced by the norms.

- A linear operator is also known as a linear transformation.
- By definition, every linear operator T is such that $T(0) = 0$.

Bounded means continuous

Theorem 3.3 – Bounded means continuous

Suppose X, Y are normed vector spaces and let $T\colon X \to Y$ be linear. Then T is continuous if and only if T is bounded.

Theorem 3.4 – Norm of an operator

Suppose X, Y are normed vector spaces. Then the set $L(X, Y)$ of all bounded, linear operators $T\colon X \to Y$ is itself a normed vector space. In fact, one may define a norm on $L(X, Y)$ by letting

$$||T|| = \sup_{\boldsymbol{x} \neq 0} \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||}.$$

- It is easy to check that $||T(\boldsymbol{x})|| \leq ||T|| \cdot ||\boldsymbol{x}||$ for all $\boldsymbol{x} \in X$.
- One also has $||S \circ T|| \leq ||S|| \cdot ||T||$ whenever $S, T \in L(X, X)$.

Norm of an operator: Example 1

- Consider the right shift operator $R\colon \ell^p \to \ell^p$ which is defined by

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

- This operator is easily seen to be linear and we also have

$$||R(\boldsymbol{x})||_p = ||\boldsymbol{x}||_p \quad \text{for all } \boldsymbol{x} \in \ell^p.$$

  In particular, the norm of this operator is equal to $||R|| = 1$.

- The left shift operator $L\colon \ell^p \to \ell^p$ is similarly defined by

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

  Since $||L(\boldsymbol{x})||_p \leq ||\boldsymbol{x}||_p$ for all $\boldsymbol{x} \in \ell^p$, we find that $||L|| \leq 1$. On the other hand, we also have $||L(\boldsymbol{x})||_p = ||\boldsymbol{x}||_p$ whenever $x_1 = 0$ and this implies that $||L|| \geq 1$. We may thus conclude that $||L|| = 1$.

Norm of an operator: Example 2

- Suppose that $T\colon (\mathbb{R}^n, ||\cdot||_1) \to (\mathbb{R}^m, ||\cdot||_\infty)$ is left multiplication by the $m \times n$ matrix A. We then have

$$||T(\boldsymbol{x})||_\infty = \max_i \left| \sum_j a_{ij} x_j \right| \leq \max_{i,j} |a_{ij}| \cdot \sum_j |x_j|$$
$$= \max_{i,j} |a_{ij}| \cdot ||\boldsymbol{x}||_1$$

  and this implies that $||T|| \leq \max_{i,j} |a_{ij}|$.
- On the other hand, the standard unit vector $\boldsymbol{x} = \boldsymbol{e}_j$ satisfies

$$\frac{||T(\boldsymbol{x})||_\infty}{||\boldsymbol{x}||_1} = \max_i \left| \sum_j a_{ij} x_j \right| = \max_i |a_{ij}|,$$

  so we also have $||T|| \geq \max_i |a_{ij}|$ for each j. We conclude that

$$||T|| = \max_{i,j} |a_{ij}|.$$

- Suppose that X is a vector space with basis $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$. Then every element $\boldsymbol{x} \in X$ can be expressed as a linear combination

$$\boldsymbol{x} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \ldots + c_k\boldsymbol{x}_k$$

for some uniquely determined coefficients $c_1, c_2, \ldots, c_k \in \mathbb{F}$.

Theorem 3.5 – Euclidean norm

Suppose that X is a vector space with basis $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$. Then one may define a norm on X using the formula

$$\boldsymbol{x} = \sum_{i=1}^{k} c_i\boldsymbol{x}_i \quad \Longrightarrow \quad ||\boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{k} |c_i|^2}.$$

This norm is also known as the Euclidean or standard norm on X.

Equivalent norms

Definition – Equivalent norms

We say that two norms $||\cdot||_a$ and $||\cdot||_b$ of a normed vector space X are equivalent, if there exist constants $C_1, C_2 > 0$ such that

$$C_1||\boldsymbol{x}||_a \leq ||\boldsymbol{x}||_b \leq C_2||\boldsymbol{x}||_a \quad \text{for all } \boldsymbol{x} \in X.$$

Theorem 3.6 – Equivalence of all norms

The norms of a finite-dimensional vector space X are all equivalent.

- The norms $||\cdot||_1$ and $||\cdot||_\infty$ are not equivalent in $C[a,b]$ because this space is complete with respect to only one of the two norms.
- In fact, $||\cdot||_p$ and $||\cdot||_q$ are not equivalent in $C[0,1]$ when $p < q$. To prove this, one may define $f_n(x) = x^n$ for each $n \in \mathbb{N}$ and then check that the quotient $||f_n||_q/||f_n||_p$ is unbounded as $n \to \infty$.

Definition – Banach space

A Banach space is a normed vector space which is also complete with respect to the metric induced by its norm.

Theorem 3.7 – Examples of Banach spaces

1. Every finite-dimensional vector space X is a Banach space.
2. The sequence space ℓ^p is a Banach space for any $1 \leq p \leq \infty$.
3. The space c_0 is a Banach space with respect to the $||\cdot||_\infty$ norm.
4. If Y is a Banach space, then $L(X, Y)$ is a Banach space.

- The space $C[a, b]$ is a Banach space with respect to the $||\cdot||_\infty$ norm. It is not complete with respect to the $||\cdot||_p$ norm when $1 \leq p < \infty$.
- Suppose that X is a Banach space and let Y be a subspace of X. Then Y is itself a Banach space if and only if Y is closed in X.

Convergence of series

Definition – Convergence of series

Suppose that $\{\boldsymbol{x}_n\}$ is a sequence in a normed vector space X. We say that the series $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ converges, if the partial sum $\boldsymbol{s}_N = \sum_{n=1}^{N} \boldsymbol{x}_n$ converges as $N \to \infty$. If that is the case, then we denote its limit by

$$\lim_{N\to\infty} \boldsymbol{s}_N = \lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n = \sum_{n=1}^{\infty} \boldsymbol{x}_n.$$

We say that $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ converges absolutely, if $\sum_{n=1}^{\infty} ||\boldsymbol{x}_n||$ converges.

Theorem 3.8 – Absolute convergence implies convergence

Suppose that X is a Banach space and let $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ be a series which converges absolutely in X. Then this series must also converge.

Invertible linear operators

Definition – Invertibility

A bounded linear operator $T\colon X \to X$ is called invertible, if there is a bounded linear operator $S\colon X \to X$ such that $S \circ T = T \circ S = I$ is the identity operator on X. If such an operator S exists, then we call it the inverse of T and we denote it by T^{-1}.

Theorem 3.9 – Geometric series

Suppose that $T\colon X \to X$ is a bounded linear operator on a Banach space X. If $||T|| < 1$, then $I - T$ is invertible with inverse $\sum_{n=0}^{\infty} T^n$.

Theorem 3.10 – Set of invertible operators

Suppose X is a Banach space. Then the set of all invertible bounded linear operators $T\colon X \to X$ is an open subset of $L(X, X)$.

Dual space

Definition – Dual space

Suppose X is a normed vector space over $\mathbb{R}$. Its dual X^* is then the set of all bounded linear operators $T\colon X \to \mathbb{R}$, namely $X^* = L(X, \mathbb{R})$.

Theorem 3.11 – Dual of $\mathbb{R}^k$

There is a bijective map $T\colon \mathbb{R}^k \to (\mathbb{R}^k)^*$ that sends each vector $\boldsymbol{a}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^k a_i x_i$.

Theorem 3.12 – Dual of ℓ^p

Suppose $1 < p < \infty$ and let $q = p/(p-1)$. Then $1/p + 1/q = 1$ and there is a bijective map $T\colon \ell^q \to (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_{\boldsymbol{a}}$ defined by $T_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{i=1}^\infty a_i x_i$.