Suppose $X$ is a vector space over the field $F = \mathbb{R}$ or $F = \mathbb{C}$. A norm on $X$ is a real-valued function $||x||$ with the following properties.

1. **Zero vector:** $||x|| = 0$ if and only if $x = 0$.
2. **Scalar factors:** $||\lambda x|| = |\lambda| \cdot ||x||$ for all $\lambda \in F$ and all $x \in X$.
3. **Triangle inequality:** $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.

A normed vector space $(X, || \cdot ||)$ consists of a vector space $X$ and a norm $||x||$. One generally thinks of $||x||$ as the length of $x$.

It is easy to check that every norm satisfies $||x|| \geq 0$ for all $x \in X$.

Every normed vector space $(X, || \cdot ||)$ is also a metric space $(X, d)$, as one may define a metric $d$ using the formula $d(x, y) = ||x - y||$. This particular metric is said to be induced by the norm.
Examples of normed vector spaces

- Given any \( p \geq 1 \), we can define a norm on \( \mathbb{R}^k \) by letting
  \[
  \|x\|_p = \left[ \sum_{i=1}^{k} |x_i|^p \right]^{1/p}.
  \]

- The space \( C[a, b] \) has a similar norm for any \( p \geq 1 \), namely
  \[
  \|f\|_p = \left[ \int_a^b |f(x)|^p \, dx \right]^{1/p}.
  \]

- There is also a norm on \( \mathbb{R}^k \) for the case \( p = \infty \). It is defined by
  \[
  \|x\|_\infty = \max_{1 \leq i \leq k} |x_i|.
  \]

- Finally, there is a similar norm on \( C[a, b] \) which is given by
  \[
  \|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.
  \]
The space $\ell^p$ consists of all real sequences $x = \{x_n\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$ 

It is a normed vector space for any $p \geq 1$ and its norm is given by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$ 

The space $\ell^\infty$ consists of all bounded real sequences $x = \{x_n\}$. It is a normed vector space and its norm is given by

$$\|x\|_\infty = \sup_{n \geq 1} |x_n|.$$ 

The space $c_0$ consists of all real sequences $\{x_n\}$ which converge to 0. It is easily seen to be a subspace of $\ell^\infty$. 
Theorem 3.1 – Product norm

Suppose $X, Y$ are normed vector spaces. Then one may define a norm on the product $X \times Y$ by letting $||(x, y)|| = ||x|| + ||y||$.

Theorem 3.2 – Continuity of operations

The following functions are continuous in any normed vector space $X$.

1. The norm $f(x) = ||x||$, where $x \in X$.
2. The vector addition $g(x, y) = x + y$, where $x, y \in X$.
3. The scalar multiplication $h(\lambda, x) = \lambda x$, where $\lambda \in \mathbb{F}$ and $x \in X$.

We shall mainly use this theorem to justify computations such as

$$\lim_{n \to \infty} ||x_n|| = \left\| \lim_{n \to \infty} x_n \right\|.$$
**Definition – Bounded, linear, continuous**

Let \( X, Y \) be normed vector spaces over the field \( F = \mathbb{R} \) or \( F = \mathbb{C} \).

1. A function \( T: X \to Y \) is called a linear operator, if
   \[
   T(x + y) = T(x) + T(y), \quad T(\lambda x) = \lambda T(x)
   \]
   for all \( x, y \in X \) and all scalars \( \lambda \in F \).

2. A function \( T: X \to Y \) is called bounded, if there exists a real number \( M > 0 \) such that \( \|T(x)\| \leq M\|x\| \) for all \( x \in X \).

3. A function \( T: X \to Y \) is called continuous, if it is continuous with respect to the metrics which are induced by the norms.

- A linear operator is also known as a linear transformation.
- By definition, every linear operator \( T \) is such that \( T(0) = 0 \).
Theorem 3.3 – Bounded means continuous

Suppose $X, Y$ are normed vector spaces and let $T : X \to Y$ be linear. Then $T$ is continuous if and only if $T$ is bounded.

Theorem 3.4 – Norm of an operator

Suppose $X, Y$ are normed vector spaces. Then the set $L(X, Y)$ of all bounded, linear operators $T : X \to Y$ is itself a normed vector space. In fact, one may define a norm on $L(X, Y)$ by letting

$$
\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.
$$

- It is easy to check that $\|T(x)\| \leq \|T\| \cdot \|x\|$ for all $x \in X$.
- One also has $\|S \circ T\| \leq \|S\| \cdot \|T\|$ whenever $S, T \in L(X, X)$. 
Consider the right shift operator $R: \ell^p \to \ell^p$ which is defined by

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

This operator is easily seen to be linear and we also have

$$\|R(x)\|_p = \|x\|_p \quad \text{for all } x \in \ell^p.$$  

In particular, the norm of this operator is equal to $\|R\| = 1$.

The left shift operator $L: \ell^p \to \ell^p$ is similarly defined by

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Since $\|L(x)\|_p \leq \|x\|_p$ for all $x \in \ell^p$, we find that $\|L\| \leq 1$. On the other hand, we also have $\|L(x)\|_p = \|x\|_p$ whenever $x_1 = 0$ and this implies that $\|L\| \geq 1$. We may thus conclude that $\|L\| = 1$. 
Suppose that $T : (\mathbb{R}^n, \| \cdot \|_1) \to (\mathbb{R}^m, \| \cdot \|_\infty)$ is left multiplication by the $m \times n$ matrix $A$. We then have

$$
\|T(x)\|_\infty = \max_i \left| \sum_j a_{ij} x_j \right| \leq \max_{i,j} |a_{ij}| \cdot \sum_j |x_j| 
$$

$$
= \max_{i,j} |a_{ij}| \cdot \|x\|_1
$$

and this implies that $\|T\| \leq \max_{i,j} |a_{ij}|$.

On the other hand, the standard unit vector $x = e_j$ satisfies

$$
\frac{\|T(x)\|_\infty}{\|x\|_1} = \max_i \left| \sum_j a_{ij} x_j \right| = \max_i |a_{ij}|
$$

so we also have $\|T\| \geq \max_i |a_{ij}|$ for each $j$. We conclude that

$$
\|T\| = \max_{i,j} |a_{ij}|.
$$
Finite-dimensional vector spaces

Suppose that $X$ is a vector space with basis $x_1, x_2, \ldots, x_k$. Then every element $x \in X$ can be expressed as a linear combination

$$x = c_1x_1 + c_2x_2 + \ldots + c_kx_k$$

for some uniquely determined coefficients $c_1, c_2, \ldots, c_k \in \mathbb{F}$.

### Theorem 3.5 – Euclidean norm

Suppose that $X$ is a vector space with basis $x_1, x_2, \ldots, x_k$. Then one may define a norm on $X$ using the formula

$$x = \sum_{i=1}^{k} c_i x_i \implies \|x\|_2 = \sqrt{\sum_{i=1}^{k} |c_i|^2}.$$

This norm is also known as the Euclidean or standard norm on $X$. 
Equivalent norms

**Definition – Equivalent norms**

We say that two norms $\| \cdot \|_a$ and $\| \cdot \|_b$ of a normed vector space $X$ are equivalent, if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \| \mathbf{x} \|_a \leq \| \mathbf{x} \|_b \leq C_2 \| \mathbf{x} \|_a \quad \text{for all } \mathbf{x} \in X.$$ 

**Theorem 3.6 – Equivalence of all norms**

The norms of a finite-dimensional vector space $X$ are all equivalent.

- The norms $\| \cdot \|_1$ and $\| \cdot \|_{\infty}$ are not equivalent in $C[a, b]$ because this space is complete with respect to only one of the two norms.

- In fact, $\| \cdot \|_p$ and $\| \cdot \|_q$ are not equivalent in $C[0, 1]$ when $p < q$. To prove this, one may define $f_n(x) = x^n$ for each $n \in \mathbb{N}$ and then check that the quotient $\| f_n \|_q / \| f_n \|_p$ is unbounded as $n \to \infty$. 

### Banach spaces

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<th><strong>Definition – Banach space</strong></th>
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<td>A Banach space is a normed vector space which is also complete with respect to the metric induced by its norm.</td>
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<th><strong>Theorem 3.7 – Examples of Banach spaces</strong></th>
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<tr>
<td>1. Every finite-dimensional vector space $X$ is a Banach space.</td>
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<td>2. The sequence space $\ell^p$ is a Banach space for any $1 \leq p \leq \infty$.</td>
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<td>3. The space $c_0$ is a Banach space with respect to the $| \cdot |_\infty$ norm.</td>
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<td>4. If $Y$ is a Banach space, then $L(X, Y)$ is a Banach space.</td>
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- The space $C[a, b]$ is a Banach space with respect to the $\| \cdot \|_\infty$ norm. It is not complete with respect to the $\| \cdot \|_p$ norm when $1 \leq p < \infty$.
- Suppose that $X$ is a Banach space and let $Y$ be a subspace of $X$. Then $Y$ is itself a Banach space if and only if $Y$ is closed in $X$.  
Convergence of series

Definition – Convergence of series

Suppose that \( \{x_n\} \) is a sequence in a normed vector space \( X \). We say that the series \( \sum_{n=1}^{\infty} x_n \) converges, if the partial sum \( s_N = \sum_{n=1}^{N} x_n \) converges as \( N \to \infty \). If that is the case, then we denote its limit by

\[
\lim_{N \to \infty} s_N = \lim_{N \to \infty} \sum_{n=1}^{N} x_n = \sum_{n=1}^{\infty} x_n.
\]

We say that \( \sum_{n=1}^{\infty} x_n \) converges absolutely, if \( \sum_{n=1}^{\infty} \|x_n\| \) converges.

Theorem 3.8 – Absolute convergence implies convergence

Suppose that \( X \) is a Banach space and let \( \sum_{n=1}^{\infty} x_n \) be a series which converges absolutely in \( X \). Then this series must also converge.
### Definition – Invertibility

A bounded linear operator $T: X \to X$ is called invertible, if there is a bounded linear operator $S: X \to X$ such that $S \circ T = T \circ S = I$ is the identity operator on $X$. If such an operator $S$ exists, then we call it the inverse of $T$ and we denote it by $T^{-1}$.

### Theorem 3.9 – Geometric series

Suppose that $T: X \to X$ is a bounded linear operator on a Banach space $X$. If $\|T\| < 1$, then $I - T$ is invertible with inverse $\sum_{n=0}^{\infty} T^n$.

### Theorem 3.10 – Set of invertible operators

Suppose $X$ is a Banach space. Then the set of all invertible bounded linear operators $T: X \to X$ is an open subset of $L(X, X)$. 
**Definition – Dual space**

Suppose $X$ is a normed vector space over $\mathbb{R}$. Its dual $X^*$ is then the set of all bounded linear operators $T : X \to \mathbb{R}$, namely $X^* = L(X, \mathbb{R})$.

**Theorem 3.11 – Dual of $\mathbb{R}^k$**

There is a bijective map $T : \mathbb{R}^k \to (\mathbb{R}^k)^*$ that sends each vector $a$ to the bounded linear operator $T_a$ defined by $T_a(x) = \sum_{i=1}^k a_i x_i$.

**Theorem 3.12 – Dual of $\ell^p$**

Suppose $1 < p < \infty$ and let $q = p/(p - 1)$. Then $1/p + 1/q = 1$ and there is a bijective map $T : \ell^q \to (\ell^p)^*$ that sends each sequence $\{a_n\}$ to the bounded linear operator $T_a$ defined by $T_a(x) = \sum_{i=1}^\infty a_i x_i$. 