Chapter 2. Topological spaces

Lecture notes for MA2223

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Topological space

Definition – Topology

A topology T on a set X is a collection of subsets of X such that

1. The topology T contains both the empty set $\varnothing$ and X.
2. Every union of elements of T belongs to T.
3. Every finite intersection of elements of T belongs to T.

- A topological space (X, T) consists of a set X and a topology T.
- Every metric space (X, d) is a topological space. In fact, one may define a topology to consist of all sets which are open in X. This particular topology is said to be induced by the metric.
- The elements of a topology are often called open. This terminology may be somewhat confusing, but it is quite standard. To say that a set U is open in a topological space (X, T) is to say that $U \in T$.

Examples of topological spaces

- The discrete topology on a set X is defined as the topology which consists of all possible subsets of X.
- The indiscrete topology on a set X is defined as the topology which consists of the subsets $\varnothing$ and X only.
- Every metric space (X, d) has a topology which is induced by its metric. It consists of all subsets of X which are open in X.

Definition – Metrisable space

A topological space (X, T) is called metrisable, if there exists a metric on X such that the topology T is induced by this metric.

- The discrete topology on X is metrisable and it is actually induced by the discrete metric. On the other hand, the indiscrete topology on X is not metrisable, if X has two or more elements.

Convergence of sequences

Definition – Convergence

Let (X, T) be a topological space. A sequence $\{x_n\}$ of points of X is said to converge to the point $x \in X$ if, given any open set U that contains x, there exists an integer N such that $x_n \in U$ for all $n \geq N$.

- When a sequence $\{x_n\}$ converges to a point x, we say that x is the limit of the sequence and we write $x_n \to x$ as $n \to \infty$ or simply

$$\lim_{n \to \infty} x_n = x.$$

- When X is a metric space, this new definition of convergence agrees with the definition of convergence in metric spaces.

Theorem 2.1 – Limits are not necessarily unique

Suppose that X has the indiscrete topology and let $x \in X$. Then the constant sequence $x_n = x$ converges to y for every $y \in X$.

Closed sets

Definition – Closed set

Suppose (X, T) is a topological space and let $A \subset X$. We say that A is closed in X, if its complement $X - A$ is open in X.

Theorem 2.2 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.
2. Both $\varnothing$ and X are closed in X.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

- We have already established these statements for metric spaces and our proofs apply almost verbatim in the case of topological spaces.

Closure of a set

Definition – Closure

Suppose (X, T) is a topological space and let $A \subset X$. The closure $\overline{A}$ of A is defined as the smallest closed set that contains A. It is thus the intersection of all closed sets that contain A.

- The interval $A = [0, 1)$ has closure $\overline{A} = [0, 1]$.
- The interval $A = (0, 1)$ has closure $\overline{A} = [0, 1]$.

Theorem 2.3 – Main facts about the closure

1. One has $A \subset \overline{A}$ for any set A.
2. If $A \subset B$, then $\overline{A} \subset \overline{B}$ as well.
3. The set A is closed if and only if $\overline{A} = A$.
4. The closure of $\overline{A}$ is itself, namely $\overline{\overline{A}} = \overline{A}$.

Interior of a set

Definition – Interior

Suppose (X, T) is a topological space and let $A \subset X$. The interior A° of A is defined as the largest open set contained in A. It is thus the union of all open sets contained in A.

- The interval $A = [0, 1]$ has interior $A^\circ = (0, 1)$.
- The interval $A = [0, 1)$ has interior $A^\circ = (0, 1)$.

Theorem 2.4 – Main facts about the interior

1. One has $A^\circ \subset A$ for any set A.
2. If $A \subset B$, then $A^\circ \subset B^\circ$ as well.
3. The set A is open if and only if $A^\circ = A$.
4. The interior of A° is itself, namely $(A^\circ)^\circ = A^\circ$.

Boundary of a set

Definition – Boundary

Suppose (X, T) is a topological space and let $A \subset X$. The boundary of A is defined as the set $\partial A = \overline{A} \cap \overline{X - A}$.

Definition – Neighbourhood

Suppose (X, T) is a topological space and let $x \in X$ be an arbitrary point. A neighbourhood of x is simply an open set that contains x.

Theorem 2.5 – Characterisation of closure/interior/boundary

Suppose (X, T) is a topological space and let $A \subset X$.

1. $x \in \overline{A} \iff$ every neighbourhood of x intersects A.
2. $x \in A^\circ \iff$ some neighbourhood of x lies within A.
3. $x \in \partial A \iff$ every neighbourhood of x intersects A and $X - A$.

Interior, closure and boundary: examples

Theorem 2.6 – Interior, closure and boundary

One has $A^\circ \cap \partial A = \varnothing$ and also $A^\circ \cup \partial A = \overline{A}$ for any set A.

Set	**Interior**	**Closure**	**Boundary**
$\{1\}$	$\varnothing$	$\{1\}$	$\{1\}$
$[0,1)$	$(0,1)$	$[0,1]$	$\{0,1\}$
$(0,1)\cup(1,2)$	$(0,1)\cup(1,2)$	$[0,2]$	$\{0,1,2\}$
$[0,1]\cup\{2\}$	$(0,1)$	$[0,1]\cup\{2\}$	$\{0,1,2\}$
$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Q}$	$\varnothing$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$

Limit points

Definition – Limit point

Let (X, T) be a topological space and let $A \subset X$. We say that x is a limit point of A if every neighbourhood of x intersects A at a point other than x.

Theorem 2.7 – Limit points and closure

Let (X, T) be a topological space and let $A \subset X$. If A' is the set of all limit points of A, then the closure of A is $\overline{A} = A \cup A'$.

- Intuitively, limit points of A are limits of sequences of points of A.
- The set $A = \{1/n : n \in \mathbb{N}\}$ has only one limit point, namely $x = 0$.
- Every point of $A = (0, 1)$ is a limit point of A, while $A' = [0, 1]$.
- A set is closed if and only if it contains its limit points.

Continuity in topological spaces

Definition – Continuity

A function $f\colon X \to Y$ between topological spaces is called continuous if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Theorem 2.8 – Composition of continuous functions

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between topological spaces. Then the composition $g \circ f\colon X \to Z$ is continuous.

Theorem 2.9 – Continuity and sequences

Let $f\colon X \to Y$ be a continuous function between topological spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Subspace topology

Definition – Subspace topology

Let (X, T) be a topological space and let $A \subset X$. Then the set

$$T' = \{U \cap A : U \in T\}$$

forms a topology on A which is known as the subspace topology.

Theorem 2.10 – Inclusion maps are continuous

Let (X, T) be a topological space and let $A \subset X$. Then the inclusion map $i \colon A \to X$ which is defined by $i(x) = x$ is continuous.

Theorem 2.11 – Restriction maps are continuous

Let $f \colon X \to Y$ be a continuous function between topological spaces and let $A \subset X$. Then the restriction map $g \colon A \to Y$ which is defined by $g(x) = f(x)$ is continuous. This map is often denoted by $g = f|_A$.

Product topology

Definition – Product topology

Given two topological spaces (X, T) and (Y, T'), we define the product topology on $X \times Y$ as the collection of all unions $\bigcup_i U_i \times V_i$, where each U_i is open in X and each V_i is open in Y.

Theorem 2.12 – Projection maps are continuous

Let (X, T) and (Y, T') be topological spaces. If $X \times Y$ is equipped with the product topology, then the projection map $p_1 \colon X \times Y \to X$ defined by $p_1(x, y) = x$ is continuous. Moreover, the same is true for the projection map $p_2 \colon X \times Y \to Y$ defined by $p_2(x, y) = y$.

Theorem 2.13 – Continuous map into a product space

Let X, Y, Z be topological spaces. Then a function $f \colon Z \to X \times Y$ is continuous if and only if its components $p_1 \circ f$, $p_2 \circ f$ are continuous.

Hausdorff spaces

Definition – Hausdorff space

We say that a topological space (X, T) is Hausdorff if any two distinct points of X have neighbourhoods which do not intersect.

- If a space X has the discrete topology, then X is Hausdorff.
- If a space X has the indiscrete topology and it contains two or more elements, then X is not Hausdorff.

Theorem 2.14 – Main facts about Hausdorff spaces

1. Every metric space is Hausdorff.
2. Every subset of a Hausdorff space is Hausdorff.
3. Every finite subset of a Hausdorff space is closed.
4. The product of two Hausdorff spaces is Hausdorff.
5. A convergent sequence in a Hausdorff space has a unique limit.

Connected spaces, part 1

Definition – Connected

Two sets A, B form a partition $A|B$ of a topological space (X, T), if they are nonempty, open and disjoint with $A \cup B = X$. We say that the space X is connected, if it has no such partition $A|B$.

Theorem 2.15 – Some facts about connected spaces

1. To say that X is connected is to say that the only subsets of X which are both open and closed in X are the subsets $\varnothing, X$.
2. The continuous image of a connected space is connected: if X is connected and $f\colon X \to Y$ is continuous, then $f(X)$ is connected.
3. A subset of $\mathbb{R}$ is connected if and only if it is an interval.
4. If a connected space A is a subset of X and the sets U, V form a partition of X, then A must lie entirely within either U or V.

Connected spaces, part 2

Theorem 2.16 – Some more facts about connected spaces

1. If A is a connected subset of X, then $\overline{A}$ is connected as well.
2. Consider a collection of connected sets U_i that have a point in common. Then the union of these sets is connected as well.
3. The product of two connected spaces is connected.

Definition – Connected component

Let (X, T) be a topological space. The connected component of a point $x \in X$ is the largest connected subset of X that contains x.

Theorem 2.17 – Connected components are closed

Let (X, T) be a topological space. Then X is the disjoint union of its connected components and each connected component is closed in X.

Compact spaces, part 1

Definition – Compactness

Let (X, T) be a topological space and let $A \subset X$. An open cover of A is a collection of open sets whose union contains A. An open subcover is a subcollection which still forms an open cover. We say that A is compact if every open cover of A has a finite subcover.

- The intervals $(-n, n)$ with $n \in \mathbb{N}$ form an open cover of $\mathbb{R}$, but this cover has no finite subcover, so $\mathbb{R}$ is not compact.
- Suppose $\{x_n\}$ is a sequence that converges to the point x. Then the set $A = \{x, x_1, x_2, x_3, \ldots\}$ is easily seen to be compact.

Theorem 2.18 – Compactness and convergence

Suppose that X is a compact metric space. Then every sequence in X has a convergent subsequence.

Compact spaces, part 2

Theorem 2.19 – Main facts about compact spaces

1. A compact subset of a Hausdorff space is closed.
2. A closed subset of a compact space is compact.
3. The interval $[a, b]$ is compact for all real numbers $a < b$.
4. The continuous image of a compact space is compact: if X is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.
5. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then f is bounded.
6. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then there exist points $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.
7. The product of two compact spaces is compact.

Theorem 2.20 – Heine-Borel theorem

A subset of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Homeomorphisms

Definition – Homeomorphism

A function $f\colon X \to Y$ between topological spaces is a homeomorphism if f is bijective, continuous and its inverse f^{-1} is continuous. When such a function exists, we say that X and Y are homeomorphic.

Theorem 2.21 – Main facts about homeomorphisms

1. Consider two homeomorphic topological spaces. If one of them is connected or compact or Hausdorff, then so is the other.
2. Suppose $f\colon X \to Y$ is bijective and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.

- Every open interval (a, b) is homeomorphic to $\mathbb{R}$. Thus, a complete space can be homeomorphic with a space which is not complete.
- There is no closed interval $[a, b]$ that is homeomorphic to $\mathbb{R}$ because the former space is compact and the latter space is not.

Uniform continuity in metric spaces

Definition – Uniformly continuous

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is uniformly continuous if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon \qquad \text{for all } x, y \in X.$$

Theorem 2.22 – Main facts about uniform continuity

1. Every Lipschitz continuous function is uniformly continuous.
2. Every uniformly continuous function is continuous.
3. When X is compact, a function $f\colon X \to Y$ is continuous on X if and only if it is uniformly continuous on X.

- $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ but not Lipschitz.
- $f(x) = 1/x$ is continuous on $(0, \infty)$ but not uniformly continuous.