A topology $T$ on a set $X$ is a collection of subsets of $X$ such that:

1. The topology $T$ contains both the empty set $\emptyset$ and $X$.
2. Every union of elements of $T$ belongs to $T$.
3. Every finite intersection of elements of $T$ belongs to $T$.

A topological space $(X, T)$ consists of a set $X$ and a topology $T$.

Every metric space $(X, d)$ is a topological space. In fact, one may define a topology to consist of all sets which are open in $X$. This particular topology is said to be induced by the metric.

The elements of a topology are often called open. This terminology may be somewhat confusing, but it is quite standard. To say that a set $U$ is open in a topological space $(X, T)$ is to say that $U \in T$. 

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- Every metric space $(X, d)$ is a topological space. In fact, one may define a topology to consist of all sets which are open in $X$. This particular topology is said to be induced by the metric.
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Topological space
Examples of topological spaces

- The discrete topology on a set $X$ is defined as the topology which consists of all possible subsets of $X$.
- The indiscrete topology on a set $X$ is defined as the topology which consists of the subsets $\emptyset$ and $X$ only.
- Every metric space $(X, d)$ has a topology which is induced by its metric. It consists of all subsets of $X$ which are open in $X$.

**Definition – Metrisable space**

A topological space $(X, T)$ is called metrisable, if there exists a metric on $X$ such that the topology $T$ is induced by this metric.

- The discrete topology on $X$ is metrisable and it is actually induced by the discrete metric. On the other hand, the indiscrete topology on $X$ is not metrisable, if $X$ has two or more elements.
**Definition – Convergence**

Let \((X, T)\) be a topological space. A sequence \(\{x_n\}\) of points of \(X\) is said to converge to the point \(x \in X\) if, given any open set \(U\) that contains \(x\), there exists an integer \(N\) such that \(x_n \in U\) for all \(n \geq N\).

- When a sequence \(\{x_n\}\) converges to a point \(x\), we say that \(x\) is the limit of the sequence and we write \(x_n \to x\) as \(n \to \infty\) or simply
  \[
  \lim_{n \to \infty} x_n = x.
  \]

- When \(X\) is a metric space, this new definition of convergence agrees with the definition of convergence in metric spaces.

**Theorem 2.1 – Limits are not necessarily unique**

Suppose that \(X\) has the indiscrete topology and let \(x \in X\). Then the constant sequence \(x_n = x\) converges to \(y\) for every \(y \in X\).
Closed sets

**Definition – Closed set**

Suppose \((X, T)\) is a topological space and let \(A \subset X\). We say that \(A\) is closed in \(X\), if its complement \(X - A\) is open in \(X\).

**Theorem 2.2 – Main facts about closed sets**

1. If a subset \(A \subset X\) is closed in \(X\), then every sequence of points of \(A\) that converges must converge to a point of \(A\).
2. Both \(\emptyset\) and \(X\) are closed in \(X\).
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

- We have already established these statements for metric spaces and our proofs apply almost verbatim in the case of topological spaces.
Closure of a set

**Definition – Closure**

Suppose $(X, T)$ is a topological space and let $A \subset X$. The closure $\overline{A}$ of $A$ is defined as the smallest closed set that contains $A$. It is thus the intersection of all closed sets that contain $A$.

- The interval $A = [0, 1)$ has closure $\overline{A} = [0, 1]$.
- The interval $A = (0, 1)$ has closure $\overline{A} = [0, 1]$.

**Theorem 2.3 – Main facts about the closure**

1. One has $A \subset \overline{A}$ for any set $A$.
2. If $A \subset B$, then $\overline{A} \subset \overline{B}$ as well.
3. The set $A$ is closed if and only if $\overline{A} = A$.
4. The closure of $\overline{A}$ is itself, namely $\overline{\overline{A}} = \overline{A}$. 
**Interior of a set**

### Definition – Interior

Suppose \((X, T)\) is a topological space and let \(A \subset X\). The interior \(A^\circ\) of \(A\) is defined as the largest open set contained in \(A\). It is thus the union of all open sets contained in \(A\).

- The interval \(A = [0, 1]\) has interior \(A^\circ = (0, 1)\).
- The interval \(A = [0, 1)\) has interior \(A^\circ = (0, 1)\).

### Theorem 2.4 – Main facts about the interior

1. One has \(A^\circ \subset A\) for any set \(A\).
2. If \(A \subset B\), then \(A^\circ \subset B^\circ\) as well.
3. The set \(A\) is open if and only if \(A^\circ = A\).
4. The interior of \(A^\circ\) is itself, namely \((A^\circ)^\circ = A^\circ\).
Boundary of a set

**Definition – Boundary**

Suppose \((X, T)\) is a topological space and let \(A \subset X\). The boundary of \(A\) is defined as the set \(\partial A = \overline{A} \cap \overline{X - A}\).  

**Definition – Neighbourhood**

Suppose \((X, T)\) is a topological space and let \(x \in X\) be an arbitrary point. A neighbourhood of \(x\) is simply an open set that contains \(x\).

**Theorem 2.5 – Characterisation of closure/interior/boundary**

Suppose \((X, T)\) is a topological space and let \(A \subset X\).

1. \(x \in \overline{A} \iff\) every neighbourhood of \(x\) intersects \(A\).
2. \(x \in \overset{\circ}{A} \iff\) some neighbourhood of \(x\) lies within \(A\).
3. \(x \in \partial A \iff\) every neighbourhood of \(x\) intersects \(A\) and \(X - A\).
Theorem 2.6 – Interior, closure and boundary

One has $A^\circ \cap \partial A = \emptyset$ and also $A^\circ \cup \partial A = \overline{A}$ for any set $A$.

<table>
<thead>
<tr>
<th>Set</th>
<th>Interior</th>
<th>Closure</th>
<th>Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1}$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$[0, 1)$</td>
<td>$(0, 1)$</td>
<td>$[0, 1]$</td>
<td>${0, 1}$</td>
</tr>
<tr>
<td>$(0, 1) \cup (1, 2)$</td>
<td>$(0, 1) \cup (1, 2)$</td>
<td>$[0, 2]$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>$[0, 1) \cup {2}$</td>
<td>$(0, 1)$</td>
<td>$[0, 1] \cup {2}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
**Definition – Limit point**

Let \((X, T)\) be a topological space and let \(A \subset X\). We say that \(x\) is a limit point of \(A\) if every neighbourhood of \(x\) intersects \(A\) at a point other than \(x\).

**Theorem 2.7 – Limit points and closure**

Let \((X, T)\) be a topological space and let \(A \subset X\). If \(A'\) is the set of all limit points of \(A\), then the closure of \(A\) is \(\overline{A} = A \cup A'\).

- Intuitively, limit points of \(A\) are limits of sequences of points of \(A\).
- The set \(A = \{1/n : n \in \mathbb{N}\}\) has only one limit point, namely \(x = 0\).
- Every point of \(A = (0, 1)\) is a limit point of \(A\), while \(A' = [0, 1]\).
- A set is closed if and only if it contains its limit points.
### Continuity in topological spaces

<table>
<thead>
<tr>
<th>Definition – Continuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function $f: X \to Y$ between topological spaces is called continuous if $f^{-1}(U)$ is open in $X$ for each set $U$ which is open in $Y$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 2.8 – Composition of continuous functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose $f: X \to Y$ and $g: Y \to Z$ are continuous functions between topological spaces. Then the composition $g \circ f: X \to Z$ is continuous.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 2.9 – Continuity and sequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $f: X \to Y$ be a continuous function between topological spaces and let ${x_n}$ be a sequence of points of $X$ which converges to $x \in X$. Then the sequence ${f(x_n)}$ must converge to $f(x)$.</td>
</tr>
</tbody>
</table>
**Subspace topology**

<table>
<thead>
<tr>
<th>Definition – Subspace topology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ((X, T)) be a topological space and let (A \subseteq X). Then the set (T' = {U \cap A : U \in T}) forms a topology on (A) which is known as the subspace topology.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 2.10 – Inclusion maps are continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ((X, T)) be a topological space and let (A \subseteq X). Then the inclusion map (i : A \rightarrow X) which is defined by (i(x) = x) is continuous.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 2.11 – Restriction maps are continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let (f : X \rightarrow Y) be a continuous function between topological spaces and let (A \subseteq X). Then the restriction map (g : A \rightarrow Y) which is defined by (g(x) = f(x)) is continuous. This map is often denoted by (g = f\big</td>
</tr>
</tbody>
</table>
**Definition – Product topology**

Given two topological spaces \((X, T)\) and \((Y, T')\), we define the product topology on \(X \times Y\) as the collection of all unions \(\bigcup_{i} U_i \times V_i\), where each \(U_i\) is open in \(X\) and each \(V_i\) is open in \(Y\).

**Theorem 2.12 – Projection maps are continuous**

Let \((X, T)\) and \((Y, T')\) be topological spaces. If \(X \times Y\) is equipped with the product topology, then the projection map \(p_1 : X \times Y \to X\) defined by \(p_1(x, y) = x\) is continuous. Moreover, the same is true for the projection map \(p_2 : X \times Y \to Y\) defined by \(p_2(x, y) = y\).

**Theorem 2.13 – Continuous map into a product space**

Let \(X, Y, Z\) be topological spaces. Then a function \(f : Z \to X \times Y\) is continuous if and only if its components \(p_1 \circ f, p_2 \circ f\) are continuous.
Hausdorff spaces

**Definition – Hausdorff space**

We say that a topological space \((X, T)\) is Hausdorff if any two distinct points of \(X\) have neighbourhoods which do not intersect.

- If a space \(X\) has the discrete topology, then \(X\) is Hausdorff.
- If a space \(X\) has the indiscrete topology and it contains two or more elements, then \(X\) is not Hausdorff.

**Theorem 2.14 – Main facts about Hausdorff spaces**

1. Every metric space is Hausdorff.
2. Every subset of a Hausdorff space is Hausdorff.
3. Every finite subset of a Hausdorff space is closed.
4. The product of two Hausdorff spaces is Hausdorff.
5. A convergent sequence in a Hausdorff space has a unique limit.
Connected spaces, part 1

**Definition – Connected**

Two sets $A, B$ form a partition $A\big|B$ of a topological space $(X, T)$, if they are nonempty, open and disjoint with $A \cup B = X$. We say that the space $X$ is connected, if it has no such partition $A\big|B$.

**Theorem 2.15 – Some facts about connected spaces**

1. To say that $X$ is connected is to say that the only subsets of $X$ which are both open and closed in $X$ are the subsets $\emptyset, X$.
2. The continuous image of a connected space is connected: if $X$ is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.
3. A subset of $\mathbb{R}$ is connected if and only if it is an interval.
4. If a connected space $A$ is a subset of $X$ and the sets $U, V$ form a partition of $X$, then $A$ must lie entirely within either $U$ or $V$. 
Theorem 2.16 – Some more facts about connected spaces

1. If $A$ is a connected subset of $X$, then $\overline{A}$ is connected as well.
2. Consider a collection of connected sets $U_i$ that have a point in common. Then the union of these sets is connected as well.
3. The product of two connected spaces is connected.

Definition – Connected component

Let $(X, T)$ be a topological space. The connected component of a point $x \in X$ is the largest connected subset of $X$ that contains $x$.

Theorem 2.17 – Connected components are closed

Let $(X, T)$ be a topological space. Then $X$ is the disjoint union of its connected components and each connected component is closed in $X$. 
Compact spaces, part 1

**Definition – Compactness**

Let \((X, T)\) be a topological space and let \(A \subset X\). An open cover of \(A\) is a collection of open sets whose union contains \(A\). An open subcover is a subcollection which still forms an open cover. We say that \(A\) is compact if every open cover of \(A\) has a finite subcover.

- The intervals \((-n, n)\) with \(n \in \mathbb{N}\) form an open cover of \(\mathbb{R}\), but this cover has no finite subcover, so \(\mathbb{R}\) is not compact.
- Suppose \(\{x_n\}\) is a sequence that converges to the point \(x\). Then the set \(A = \{x, x_1, x_2, x_3, \ldots\}\) is easily seen to be compact.

**Theorem 2.18 – Compactness and convergence**

Suppose that \(X\) is a compact metric space. Then every sequence in \(X\) has a convergent subsequence.
## Theorem 2.19 – Main facts about compact spaces

1. A compact subset of a Hausdorff space is closed.
2. A closed subset of a compact space is compact.
3. The interval \([a, b]\) is compact for all real numbers \(a < b\).
4. The continuous image of a compact space is compact: if \(X\) is compact and \(f : X \to Y\) is continuous, then \(f(X)\) is compact.
5. If \(X\) is compact and \(f : X \to \mathbb{R}\) is continuous, then \(f\) is bounded.
6. If \(X\) is compact and \(f : X \to \mathbb{R}\) is continuous, then there exist points \(a, b \in X\) such that \(f(a) \leq f(x) \leq f(b)\) for all \(x \in X\).
7. The product of two compact spaces is compact.

## Theorem 2.20 – Heine-Borel theorem

A subset of \(\mathbb{R}^k\) is compact if and only if it is closed and bounded.
**Definition – Homeomorphism**

A function \( f : X \rightarrow Y \) between topological spaces is a homeomorphism if \( f \) is bijective, continuous and its inverse \( f^{-1} \) is continuous. When such a function exists, we say that \( X \) and \( Y \) are homeomorphic.

**Theorem 2.21 – Main facts about homeomorphisms**

1. Consider two homeomorphic topological spaces. If one of them is connected or compact or Hausdorff, then so is the other.

2. Suppose \( f : X \rightarrow Y \) is bijective and continuous. If \( X \) is compact and \( Y \) is Hausdorff, then \( f \) is a homeomorphism.

- Every open interval \((a, b)\) is homeomorphic to \( \mathbb{R} \). Thus, a complete space can be homeomorphic with a space which is not complete.
- There is no closed interval \([a, b]\) that is homeomorphic to \( \mathbb{R} \) because the former space is compact and the latter space is not.
Definition – Uniformly continuous

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function \(f : X \rightarrow Y\) is uniformly continuous if, given any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon\]

for all \(x, y \in X\).

Theorem 2.22 – Main facts about uniform continuity

1. Every Lipschitz continuous function is uniformly continuous.
2. Every uniformly continuous function is continuous.
3. When \(X\) is compact, a function \(f : X \rightarrow Y\) is continuous on \(X\) if and only if it is uniformly continuous on \(X\).

- \(f(x) = \sqrt{x}\) is uniformly continuous on \([0, 1]\) but not Lipschitz.
- \(f(x) = \frac{1}{x}\) is continuous on \((0, \infty)\) but not uniformly continuous.