Chapter 1. Metric spaces

Lecture notes for MA2223

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The definition of a metric

Definition – Metric

A metric on a set X is a function d that assigns a real number to each pair of elements of X in such a way that the following properties hold.

1. Non-negativity: $d(x, y) \geq 0$ with equality if and only if $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

- A metric space is a pair (X, d), where X is a set and d is a metric defined on X. The metric is often regarded as a distance function.
- The usual metric on $\mathbb{R}$ is the one given by $d(x, y) = |x - y|$.
- A metric can be used to define limits and continuity of functions. In fact, the ε-δ definition for functions on $\mathbb{R}$ can be easily adjusted so that it applies to functions on an arbitrary metric space.

Examples of metrics in $\mathbb{R}^k$

- The usual metric in $\mathbb{R}^k$ is the Euclidean metric d_2 defined by

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{k} |x_i - y_i|^2 \right]^{1/2}.$$

- The metric d_1 is defined using the formula

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{k} |x_i - y_i|.$$

- One may define a metric d_p for each $p \geq 1$ by setting

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{k} |x_i - y_i|^p \right]^{1/p}.$$

- Finally, there is a metric d_∞ which is defined by

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Examples of other metrics

- The discrete metric on a nonempty set X is defined by letting

$$d(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{array} \right\}.$$

- Let $C[a,b]$ denote the set of all continuous functions $f\colon [a,b] \to \mathbb{R}$. A metric on $C[a,b]$ is then given by the formula

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

- Another metric on $C[a,b]$ is given by the formula

$$d_\infty(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|.$$

  Here, the supremum could also be replaced by a maximum.

Technical inequalities

Theorem 1.1 – Technical inequalities

Suppose that $x, y \geq 0$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be arbitrary vectors in $\mathbb{R}^k$.

1. **Young's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

2. **Hölder's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{k} |a_i| \cdot |b_i| \leq \left[\sum_{i=1}^{k} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{k} |b_i|^q\right]^{1/q}.$$

3. **Minkowski's inequality**: If $p > 1$, then

$$d_p(\boldsymbol{a}, \boldsymbol{b}) \leq d_p(\boldsymbol{a}, \boldsymbol{c}) + d_p(\boldsymbol{c}, \boldsymbol{b}).$$

Open balls

Definition – Open ball

Suppose (X, d) is a metric space and let $x \in X$ be an arbitrary point. The open ball with centre x and radius $r > 0$ is defined as

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

- The open balls in $\mathbb{R}$ are the open intervals $B(x, r) = (x - r, x + r)$. The open interval (a, b) has centre $(b + a)/2$ and radius $(b - a)/2$.
- If the metric on X is discrete, then $B(x, 1) = \{x\}$ for all $x \in X$.
- The open ball $B(0, 1)$ in $X = [0, 2]$ is given by $B(0, 1) = [0, 1)$.

Definition – Bounded

Let (X, d) be a metric space and $A \subset X$. We say that A is bounded, if there exist a point $x \in X$ and some $r > 0$ such that $A \subset B(x, r)$.

Open sets

Definition – Open set

Given a metric space (X, d), we say that a subset $U \subset X$ is open in X if, for each point $x \in U$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. In other words, each $x \in U$ is the centre of an open ball that lies in U.

Theorem 1.2 – Main facts about open sets

1. If X is a metric space, then both $\varnothing$ and X are open in X.
2. Arbitrary unions of open sets are open.
3. Finite intersections of open sets are open.
4. Every open ball is an open set.
5. A set is open if and only if it is a union of open balls.

- Infinite intersections of open sets are not necessarily open.
- If the metric on X is discrete, then every subset of X is open in X.

Convergence of sequences

Definition – Convergence

Let (X, d) be a metric space. We say that a sequence $\{x_n\}$ of points of X converges to the point $x \in X$ if, given any $\varepsilon > 0$ there exists an integer N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

- When a sequence $\{x_n\}$ converges to a point x, we say that x is the limit of the sequence and we write $x_n \to x$ as $n \to \infty$ or simply

$$\lim_{n\to\infty} x_n = x.$$

- A sequence $\boldsymbol{x}_n = (x_{n1}, x_{n2}, \dots, x_{nk})$ of points in $\mathbb{R}^k$ converges if and only if each of the components x_{ni} converges in $\mathbb{R}$.

Theorem 1.3 – Limits are unique

The limit of a sequence in a metric space is unique. In other words, no sequence may converge to two different limits.

Closed sets

Definition – Closed set

Suppose (X, d) is a metric space and let $A \subset X$. We say that A is closed in X, if its complement $X - A$ is open in X.

Theorem 1.4 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.
2. Both $\varnothing$ and X are closed in X.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

- The last two statements can be established using De Morgan's laws

$$X - \bigcup_i U_i = \bigcap_i (X - U_i), \qquad X - \bigcap_i U_i = \bigcup_i (X - U_i).$$

Continuity in metric spaces

Definition – Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is continuous at $x \in X$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \quad\Longrightarrow\quad d_Y(f(x), f(y)) < \varepsilon.$$

We also say that f is continuous, if f is continuous at all points.

- One may express the above definition in terms of open balls as

$$y \in B(x, \delta) \quad\Longrightarrow\quad f(y) \in B(f(x), \varepsilon).$$

- If $f\colon X \to Y$ is a constant function, then f is continuous.
- Every function $f\colon X \to Y$ is continuous, if d_X is discrete.

Theorems involving continuity

Theorem 1.5 – Composition of continuous functions

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between metric spaces. Then the composition $g \circ f\colon X \to Z$ is continuous.

Theorem 1.6 – Continuity and sequences

Suppose $f\colon X \to Y$ is a continuous function between metric spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Theorem 1.7 – Continuity and open sets

A function $f\colon X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Lipschitz continuity

Definition – Lipschitz continuous

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is Lipschitz continuous, if there is a constant $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) \quad \text{for all } x, y \in X.$$

Theorem 1.8 – Main facts about Lipschitz continuity

1. Every Lipschitz continuous function is continuous.
2. If a function $f\colon [a, b] \to \mathbb{R}$ is differentiable and its derivative is bounded, then f is Lipschitz continuous on $[a, b]$.

- The function $f(x) = x^2$ is Lipschitz continuous on $[0, 1]$.
- The function $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$.

Convergence of functions

Definition – Pointwise and uniform convergence

Let $\{f_n(x)\}$ be a sequence of functions $f_n \colon X \to \mathbb{R}$, where X is a metric space. We say that $f_n(x)$ converges pointwise to $f(x)$ if, given any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N.$$

We also say that f_n converges to f uniformly on X if, given any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and all } x \in X.$$

- For pointwise convergence, one gets to choose N depending on x.
- For uniform convergence, the same choice of N should work for all x.
- If a sequence converges uniformly, then it also converges pointwise.

Pointwise and uniform convergence

Theorem 1.9 – Pointwise and uniform convergence

1. To say that $f_n(x) \to f(x)$ pointwise is to say that

$$|f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

2. To say that $f_n \to f$ uniformly on X is to say that

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Theorem 1.10 – Uniform limit of continuous functions

The uniform limit of continuous functions is continuous: if each f_n is continuous and $f_n \to f$ uniformly on X, then f is continuous on X.

- The pointwise limit of continuous functions need not be continuous. For instance, x^n converges to 0 if $0 \leq x < 1$ and to 1 if $x = 1$.

Cauchy sequences

Definition – Cauchy sequence

Let (X,d) be a metric space. A sequence $\{x_n\}$ of points of X is called Cauchy if, given any $\varepsilon > 0$ there exists an integer N such that

$$d(x_m, x_n) < \varepsilon \quad \text{for all } m, n \geq N.$$

Theorem 1.11 – Convergent implies Cauchy

In a metric space, every convergent sequence is a Cauchy sequence.

Theorem 1.12 – Cauchy implies bounded

In a metric space, every Cauchy sequence is bounded.

- A Cauchy sequence does not have to be convergent. For instance, the sequence $x_n = 1/n$ is Cauchy but not convergent in $X = (0,2)$.

Completeness of $\mathbb{R}$

Definition – Complete metric space

A metric space (X, d) is called complete if every Cauchy sequence of points of X actually converges to a point of X.

Theorem 1.13 – Cauchy sequence with convergent subsequence

Suppose (X, d) is a metric space and let $\{x_n\}$ be a Cauchy sequence in X that has a convergent subsequence. Then $\{x_n\}$ converges itself.

Theorem 1.14 – Completeness of $\mathbb{R}$

1. Every sequence in $\mathbb{R}$ which is monotonic and bounded converges.
2. **Bolzano-Weierstrass theorem**: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
3. The set $\mathbb{R}$ of all real numbers is a complete metric space.

Completeness

Theorem 1.15 – Examples of complete metric spaces

1. The space $\mathbb{R}^k$ is complete with respect to its usual metric.
2. The space $C[a,b]$ is complete with respect to the d_∞ metric.

- The space $\mathbb{R}^k$ is complete with respect to any d_p metric. One can prove this fact by noting that $d_\infty(\boldsymbol{x},\boldsymbol{y}) \leq d_p(\boldsymbol{x},\boldsymbol{y}) \leq k^{1/p} d_\infty(\boldsymbol{x},\boldsymbol{y})$.
- The space $C[a,b]$ is not complete with respect to the d_1 metric. One can find Cauchy sequences that converge to a discontinuous function.
- The set $A = \{1/n : n \in \mathbb{N}\}$ is not complete. It contains a sequence which converges in $\mathbb{R}$, but this sequence does not converge in A.

Theorem 1.16 – Subsets of a complete metric space

Suppose (X,d) is a complete metric space and let $A \subset X$. Then A is complete if and only if A is closed in X.

Banach's fixed point theorem

Definition – Contraction

Let (X, d) be a metric space. We say that a function $f\colon X \to X$ is a contraction, if there exists a constant $0 \leq \alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 1.17 – Banach's fixed point theorem

If $f\colon X \to X$ is a contraction on a complete metric space X, then f has a unique fixed point, namely a unique point x with $f(x) = x$.

- Every contraction is Lipschitz continuous, hence also continuous.
- Consider the function $f\colon (0, 1) \to (0, 1)$ defined by $f(x) = x/2$. This is easily seen to be a contraction, but it has no fixed point on $(0, 1)$. Thus, one does need X to be complete for the theorem to hold.

Theorem 1.18 – Existence and uniqueness of solutions

Consider an initial value problem of the form

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

If f is continuous in t and Lipschitz continuous in y, then there exists a unique solution $y(t)$ which is defined on $[0, \varepsilon]$ for some $\varepsilon > 0$.

- To say that $y(t)$ is a solution is to say that $y(t)$ is a fixed point of

$$\mathcal{A}(y(t)) = y_0 + \int_0^t f(s, y(s))\, ds.$$

- In general, solutions of differential equations need not be defined for all times. For instance, $y(t) = 1/(1-t)$ is the unique solution of

$$y'(t) = y(t)^2, \qquad y(0) = 1.$$

This solution is defined at time $t = 0$ but not at time $t = 1$.

Completion of a metric space

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f\colon X \to X'$ such that X' is complete.

- A distance preserving map is called an isometry, while X' is called a completion of X. It is easy to check that every distance preserving map is injective. Thus, one can always regard X as a subset of X'.
- The proof of this theorem is somewhat long, but the general idea is to define a relation on the set of Cauchy sequences in X by letting

$$\{x_n\} \sim \{y_n\} \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

This turns out to be an equivalence relation and the completion X' is the set of all equivalence classes with metric d' defined by

$$d'([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n).$$