Chapter 1. Metric spaces
Lecture notes for MA2223

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The definition of a metric

**Definition – Metric**

A metric on a set $X$ is a function $d$ that assigns a real number to each pair of elements of $X$ in such a way that the following properties hold.

1. **Non-negativity:** $d(x, y) \geq 0$ with equality if and only if $x = y$.
2. **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. **Triangle inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

- A metric space is a pair $(X, d)$, where $X$ is a set and $d$ is a metric defined on $X$. The metric is often regarded as a distance function.
- The usual metric on $\mathbb{R}$ is the one given by $d(x, y) = |x - y|$.
- A metric can be used to define limits and continuity of functions. In fact, the $\varepsilon$-$\delta$ definition for functions on $\mathbb{R}$ can be easily adjusted so that it applies to functions on an arbitrary metric space.
Examples of metrics in $\mathbb{R}^k$

- The usual metric in $\mathbb{R}^k$ is the Euclidean metric $d_2$ defined by
  \[
d_2(x, y) = \left[ \sum_{i=1}^{k} |x_i - y_i|^2 \right]^{1/2}.
  \]

- The metric $d_1$ is defined using the formula
  \[
d_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|.
  \]

- One may define a metric $d_p$ for each $p \geq 1$ by setting
  \[
d_p(x, y) = \left[ \sum_{i=1}^{k} |x_i - y_i|^p \right]^{1/p}.
  \]

- Finally, there is a metric $d_\infty$ which is defined by
  \[
d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|.
  \]
Examples of other metrics

- The discrete metric on a nonempty set $X$ is defined by letting

$$d(x, y) = \begin{cases} 
1 & \text{if } x \neq y \\
0 & \text{if } x = y 
\end{cases}.$$ 

- Let $C[a, b]$ denote the set of all continuous functions $f: [a, b] \rightarrow \mathbb{R}$. A metric on $C[a, b]$ is then given by the formula

$$d_1(f, g) = \int_a^b |f(x) - g(x)| \, dx.$$ 

- Another metric on $C[a, b]$ is given by the formula

$$d_\infty(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)|.$$ 

Here, the supremum could also be replaced by a maximum.
Suppose that $x, y \geq 0$ and let $a, b, c$ be arbitrary vectors in $\mathbb{R}^k$.

**Young’s inequality:** If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$ 

**Hölder’s inequality:** If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{k} |a_i| \cdot |b_i| \leq \left[ \sum_{i=1}^{k} |a_i|^p \right]^{1/p} \left[ \sum_{i=1}^{k} |b_i|^q \right]^{1/q}.$$ 

**Minkowski’s inequality:** If $p > 1$, then

$$d_p(a, b) \leq d_p(a, c) + d_p(c, b).$$
Open balls

**Definition – Open ball**

Suppose $(X, d)$ is a metric space and let $x \in X$ be an arbitrary point. The open ball with centre $x$ and radius $r > 0$ is defined as

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

- The open balls in $\mathbb{R}$ are the open intervals $B(x, r) = (x - r, x + r)$.
- The open interval $(a, b)$ has centre $(b + a)/2$ and radius $(b - a)/2$.
- If the metric on $X$ is discrete, then $B(x, 1) = \{x\}$ for all $x \in X$.
- The open ball $B(0, 1)$ in $X = [0, 2]$ is given by $B(0, 1) = [0, 1)$.

**Definition – Bounded**

Let $(X, d)$ be a metric space and $A \subset X$. We say that $A$ is bounded, if there exist a point $x \in X$ and some $r > 0$ such that $A \subset B(x, r)$. 
Definition – Open set

Given a metric space \((X, d)\), we say that a subset \(U \subset X\) is open in \(X\) if, for each point \(x \in U\) there exists \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subset U\). In other words, each \(x \in U\) is the centre of an open ball that lies in \(U\).

Theorem 1.2 – Main facts about open sets

1. If \(X\) is a metric space, then both \(\emptyset\) and \(X\) are open in \(X\).
2. Arbitrary unions of open sets are open.
3. Finite intersections of open sets are open.
4. Every open ball is an open set.
5. A set is open if and only if it is a union of open balls.

- Infinite intersections of open sets are not necessarily open.
- If the metric on \(X\) is discrete, then every subset of \(X\) is open in \(X\).
Convergence of sequences

**Definition – Convergence**

Let \((X, d)\) be a metric space. We say that a sequence \(\{x_n\}\) of points of \(X\) converges to the point \(x \in X\) if, given any \(\varepsilon > 0\) there exists an integer \(N\) such that \(d(x_n, x) < \varepsilon\) for all \(n \geq N\).

- When a sequence \(\{x_n\}\) converges to a point \(x\), we say that \(x\) is the limit of the sequence and we write \(x_n \to x\) as \(n \to \infty\) or simply \(\lim_{n \to \infty} x_n = x\).

- A sequence \(x_n = (x_{n1}, x_{n2}, \ldots, x_{nk})\) of points in \(\mathbb{R}^k\) converges if and only if each of the components \(x_{ni}\) converges in \(\mathbb{R}\).

**Theorem 1.3 – Limits are unique**

The limit of a sequence in a metric space is unique. In other words, no sequence may converge to two different limits.
Closed sets

**Definition – Closed set**

Suppose $(X, d)$ is a metric space and let $A \subset X$. We say that $A$ is closed in $X$, if its complement $X - A$ is open in $X$.

**Theorem 1.4 – Main facts about closed sets**

1. If a subset $A \subset X$ is closed in $X$, then every sequence of points of $A$ that converges must converge to a point of $A$.
2. Both $\emptyset$ and $X$ are closed in $X$.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

The last two statements can be established using De Morgan’s laws

$$X - \bigcup_{i} U_i = \bigcap_{i} (X - U_i), \quad X - \bigcap_{i} U_i = \bigcup_{i} (X - U_i).$$
Continuity in metric spaces

**Definition – Continuity**

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function \(f : X \to Y\) is continuous at \(x \in X\) if, given any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.
\]

We also say that \(f\) is continuous, if \(f\) is continuous at all points.

- One may express the above definition in terms of open balls as
  \[
y \in B(x, \delta) \implies f(y) \in B(f(x), \varepsilon).
  \]

- If \(f : X \to Y\) is a constant function, then \(f\) is continuous.
- Every function \(f : X \to Y\) is continuous, if \(d_X\) is discrete.
<table>
<thead>
<tr>
<th>Theorem 1.5 – Composition of continuous functions</th>
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<tbody>
<tr>
<td>Suppose ( f: X \to Y ) and ( g: Y \to Z ) are continuous functions between metric spaces. Then the composition ( g \circ f: X \to Z ) is continuous.</td>
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<th>Theorem 1.6 – Continuity and sequences</th>
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<td>Suppose ( f: X \to Y ) is a continuous function between metric spaces and let ( { x_n } ) be a sequence of points of ( X ) which converges to ( x \in X ). Then the sequence ( { f(x_n) } ) must converge to ( f(x) ).</td>
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<th>Theorem 1.7 – Continuity and open sets</th>
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<td>A function ( f: X \to Y ) between metric spaces is continuous if and only if ( f^{-1}(U) ) is open in ( X ) for each set ( U ) which is open in ( Y ).</td>
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Lipschitz continuity

**Definition – Lipschitz continuous**

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function \(f: X \to Y\) is **Lipschitz continuous**, if there is a constant \(L \geq 0\) such that

\[
d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)
\]

for all \(x, y \in X\).

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**Theorem 1.8 – Main facts about Lipschitz continuity**

1. Every Lipschitz continuous function is continuous.
2. If a function \(f: [a, b] \to \mathbb{R}\) is differentiable and its derivative is bounded, then \(f\) is Lipschitz continuous on \([a, b]\).

- The function \(f(x) = x^2\) is Lipschitz continuous on \([0, 1]\).
- The function \(f(x) = \sqrt{x}\) is not Lipschitz continuous on \([0, 1]\).
Definition – Pointwise and uniform convergence

Let \( \{f_n(x)\} \) be a sequence of functions \( f_n : X \to \mathbb{R} \), where \( X \) is a metric space. We say that \( f_n(x) \) converges pointwise to \( f(x) \) if, given any \( \varepsilon > 0 \) there exists an integer \( N \) such that

\[
|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N.
\]

We also say that \( f_n \) converges to \( f \) uniformly on \( X \) if, given any \( \varepsilon > 0 \) there exists an integer \( N \) such that

\[
|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and all } x \in X.
\]

- For pointwise convergence, one gets to choose \( N \) depending on \( x \).
- For uniform convergence, the same choice of \( N \) should work for all \( x \).
- If a sequence converges uniformly, then it also converges pointwise.
Pointwise and uniform convergence

Theorem 1.9 – Pointwise and uniform convergence

1. To say that $f_n(x) \to f(x)$ pointwise is to say that

$$|f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$ 

2. To say that $f_n \to f$ uniformly on $X$ is to say that

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$ 

Theorem 1.10 – Uniform limit of continuous functions

The uniform limit of continuous functions is continuous: if each $f_n$ is continuous and $f_n \to f$ uniformly on $X$, then $f$ is continuous on $X$.

The pointwise limit of continuous functions need not be continuous. For instance, $x^n$ converges to 0 if $0 \leq x < 1$ and to 1 if $x = 1$. 

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Cauchy sequences

**Definition – Cauchy sequence**

Let \((X, d)\) be a metric space. A sequence \(\{x_n\}\) of points of \(X\) is called Cauchy if, given any \(\varepsilon > 0\) there exists an integer \(N\) such that

\[d(x_m, x_n) < \varepsilon \quad \text{for all} \quad m, n \geq N.\]

**Theorem 1.11 – Convergent implies Cauchy**

In a metric space, every convergent sequence is a Cauchy sequence.

**Theorem 1.12 – Cauchy implies bounded**

In a metric space, every Cauchy sequence is bounded.

- A Cauchy sequence does not have to be convergent. For instance, the sequence \(x_n = 1/n\) is Cauchy but not convergent in \(X = (0, 2)\).
Completeness of $\mathbb{R}$

**Definition – Complete metric space**

A metric space $(X, d)$ is called complete if every Cauchy sequence of points of $X$ actually converges to a point of $X$.

**Theorem 1.13 – Cauchy sequence with convergent subsequence**

Suppose $(X, d)$ is a metric space and let $\{x_n\}$ be a Cauchy sequence in $X$ that has a convergent subsequence. Then $\{x_n\}$ converges itself.

**Theorem 1.14 – Completeness of $\mathbb{R}$**

1. Every sequence in $\mathbb{R}$ which is monotonic and bounded converges.
2. **Bolzano-Weierstrass theorem**: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
3. The set $\mathbb{R}$ of all real numbers is a complete metric space.
Completeness

**Theorem 1.15 – Examples of complete metric spaces**

1. The space $\mathbb{R}^k$ is complete with respect to its usual metric.
2. The space $C[a, b]$ is complete with respect to the $d_\infty$ metric.

- The space $\mathbb{R}^k$ is complete with respect to any $d_p$ metric. One can prove this fact by noting that $d_\infty(x, y) \leq d_p(x, y) \leq k^{1/p}d_\infty(x, y)$.
- The space $C[a, b]$ is not complete with respect to the $d_1$ metric. One can find Cauchy sequences that converge to a discontinuous function.
- The set $A = \{1/n : n \in \mathbb{N}\}$ is not complete. It contains a sequence which converges in $\mathbb{R}$, but this sequence does not converge in $A$.

**Theorem 1.16 – Subsets of a complete metric space**

Suppose $(X, d)$ is a complete metric space and let $A \subset X$. Then $A$ is complete if and only if $A$ is closed in $X$. 
Banach’s fixed point theorem

**Definition – Contraction**

Let \((X, d)\) be a metric space. We say that a function \(f : X \to X\) is a contraction, if there exists a constant \(0 \leq \alpha < 1\) such that

\[
d(f(x), f(y)) \leq \alpha \cdot d(x, y)
\]

for all \(x, y \in X\).

**Theorem 1.17 – Banach’s fixed point theorem**

If \(f : X \to X\) is a contraction on a complete metric space \(X\), then \(f\) has a unique fixed point, namely a unique point \(x\) with \(f(x) = x\).

- Every contraction is Lipschitz continuous, hence also continuous.
- Consider the function \(f : (0, 1) \to (0, 1)\) defined by \(f(x) = x/2\). This is easily seen to be a contraction, but it has no fixed point on \((0, 1)\). Thus, one does need \(X\) to be complete for the theorem to hold.
Consider an initial value problem of the form

\[ y'(t) = f(t, y(t)), \quad y(0) = y_0. \]

If \( f \) is continuous in \( t \) and Lipschitz continuous in \( y \), then there exists a unique solution \( y(t) \) which is defined on \([0, \varepsilon]\) for some \( \varepsilon > 0 \).

- To say that \( y(t) \) is a solution is to say that \( y(t) \) is a fixed point of

\[ A(y(t)) = y_0 + \int_0^t f(s, y(s)) \, ds. \]

- In general, solutions of differential equations need not be defined for all times. For instance, \( y(t) = 1/(1 - t) \) is the unique solution of

\[ y'(t) = y(t)^2, \quad y(0) = 1. \]

This solution is defined at time \( t = 0 \) but not at time \( t = 1 \).
Theorem 1.19 – Completion of a metric space

Given a metric space \((X, d)\), there exist a metric space \((X', d')\) and a distance preserving map \(f: X \rightarrow X'\) such that \(X'\) is complete.

- A distance preserving map is called an isometry, while \(X'\) is called a completion of \(X\). It is easy to check that every distance preserving map is injective. Thus, one can always regard \(X\) as a subset of \(X'\).
- The proof of this theorem is somewhat long, but the general idea is to define a relation on the set of Cauchy sequences in \(X\) by letting
  \[
  \{x_n\} \sim \{y_n\} \iff \lim_{n \to \infty} d(x_n, y_n) = 0.
  \]
  This turns out to be an equivalence relation and the completion \(X'\) is the set of all equivalence classes with metric \(d'\) defined by
  \[
  d'([x_n], [y_n]) = \lim_{n \to \infty} d(x_n, y_n).
  \]