Homework 9. Solutions

1. Are any of the following sets homeomorphic? Explain.

$$A = (0, 1), \qquad B = [0, 1), \qquad C = [0, 1], \qquad D = [0, \infty).$$

The set C is not homeomorphic to any of the other sets because C is compact and the other sets are not. To show that B and D are homeomorphic, we note that either of the functions

$$f(x) = x/(1 - x), \qquad g(x) = \tan(\pi x/2)$$

gives rise to a homeomorphism between $B = [0, 1)$ and $D = [0, \infty)$.

Finally, we turn to A and B. Were these sets homeomorphic, we would have a homeomorphism $h\colon [0, 1) \to (0, 1)$ and its restriction on $(0, 1)$ would also be a homeomorphism. This is not possible, as the image of the restriction is $(0, 1) - \{h(0)\}$ which is not connected.

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2. Let (X, d) be a metric space and fix some $y \in X$. Show that the function $f \colon X \to \mathbb{R}$ defined by $f(x) = d(x, y)$ is Lipschitz continuous.

Letting $x, z \in X$ be arbitrary, we use the triangle inequality to get

$$\begin{aligned} f(x) = d(x,y) \leq d(x,z) + d(z,y) = d(x,z) + f(z) \\ f(z) = d(z,y) \leq d(z,x) + d(x,y) = d(x,z) + f(x). \end{aligned}$$

Once we now combine these equations, we may conclude that

$$|f(x) - f(z)| \leq d(x,z).$$

This shows that the function $f \colon X \to \mathbb{R}$ is Lipschitz continuous.

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3. Let $\boldsymbol{x}_n \in \ell^p$ denote the sequence whose first n^2 entries are equal to $1/n$ and all other entries are zero. For which values of $1 \leq p \leq \infty$ does this sequence converge to the zero sequence in ℓ^p?

Using the definition of the norm in ℓ^p, we find that

$$||\boldsymbol{x}_n - 0||_p^p = \sum_{i=1}^{\infty} |x_{ni}|^p = \sum_{i=1}^{n^2} \frac{1}{n^p} = n^{2-p}.$$

This expression converges to zero if and only if the exponent $2 - p$ is negative, hence if and only if $p > 2$.

4. Let $\boldsymbol{e}_n \in \ell^\infty$ denote the sequence whose nth entry is equal to 1 and all other entries are zero. Show that $\{\boldsymbol{e}_n\}_{n=1}^\infty$ is bounded but not Cauchy and that the unit ball $B = \{\boldsymbol{x} \in \ell^\infty : ||\boldsymbol{x}||_\infty \leq 1\}$ is closed and bounded, but not compact.

First of all, $\{\boldsymbol{e}_n\}_{n=1}^\infty$ is bounded but not Cauchy because

$$||\boldsymbol{e}_n||_\infty = ||\boldsymbol{e}_m - \boldsymbol{e}_n||_\infty = 1 \tag{$*$}$$

whenever $m \neq n$. It is clear that B is bounded. To show that B is also closed, we note that the norm $f \colon X \to \mathbb{R}$ is continuous in any normed vector space and that B is the inverse image of $(-\infty, 1]$.

Finally, suppose that B is compact. Then the sequence $\{\boldsymbol{e}_n\}$ has a convergent subsequence by Theorem 2.18. Such a subsequence is actually Cauchy and this contradicts equation $(*)$.