Homework 5. Solutions

1. Let (X, T) be a topological space and let A, B be subsets of X. Show that the closure of their union is given by $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Since $\overline{A} \cup \overline{B}$ is a closed set that contains $A \cup B$ and $\overline{A \cup B}$ is the smallest closed set that contains $A \cup B$, we must certainly have

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}.$$

To prove the opposite inclusion, we note that Theorem 2.5 gives

$$\begin{aligned} x \in \overline{A} \cup \overline{B} &\implies \text{every neighbourhood of } x \text{ intersects } A \text{ or } B \\ &\implies \text{every neighbourhood of } x \text{ intersects } A \cup B \\ &\implies x \in \overline{A \cup B}. \end{aligned}$$

2. Find two open intervals $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Pick any real numbers $a < b < c$ and consider the open intervals

$$A = (a, b), \qquad B = (b, c).$$

Since $A \cap B = \varnothing$, one also has $\overline{A \cap B} = \varnothing$. On the other hand,

$$\overline{A} \cap \overline{B} = [a, b] \cap [b, c] = \{b\}.$$

Homework 5. Solutions

3. Let (X, T) be a topological space and let $A \subset X$. Show that

$$\partial A = \varnothing \quad \iff \quad A \text{ is both open and closed in } X.$$

If A is both open and closed in X, then the boundary of A is

$$\partial A = \overline{A} \cap \overline{X - A} = A \cap (X - A) = \varnothing.$$

Conversely, suppose that $\partial A = \varnothing$. Then Theorem 2.6 gives

$$A^\circ = \overline{A}.$$

Since $A^\circ \subset A \subset \overline{A}$ by definition, these three sets are equal, so

$$A^\circ = A = \overline{A} \implies A \text{ is both open and closed in } X.$$

Homework 5. Solutions

4. Let (X, T) be a topological space and let $A \subset X$. Show that

$$\overline{X - A} = X - A^\circ.$$

Using Theorem 2.5, one finds that

$$\begin{aligned} x \in \overline{X - A} &\iff \text{every neighbourhood of } x \text{ intersects } X - A \\ &\iff \text{no neighbourhood of } x \text{ is contained in } A \\ &\iff x \notin A^\circ \\ &\iff x \in X - A^\circ. \end{aligned}$$

Homework 6. Solutions

1. Let (Y, T) be a topological space and let $A \subset Y$. Show that A is open in Y if and only if every point of A has a neighbourhood which lies within A. Hint: We know that A is open if and only if $A = A^\circ$.

As we already know, a set A is open if and only if $A = A^\circ$. On the other hand, one always has $A^\circ \subset A$ by definition, so a set A is open if and only if $A \subset A^\circ$. This is true precisely when every point of A has a neighbourhood which lies within A.

Homework 6. Solutions

2. Let (X, T) be a Hausdorff space. Show that the set

$$A = \{(x, y) \in X \times X : x \neq y\}$$

is open in $X \times X$. Hint: Use the previous problem with $Y = X \times X$.

Let (x, y) be an arbitrary point of A. Then $x \neq y$ and there exist sets U, V which are open in X with $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Now, the product $U \times V$ is a neighbourhood of (x, y) such that

$$\begin{aligned} (a, b) \in U \times V &\implies a \in U \text{ and } b \in V \\ &\implies a \neq b \\ &\implies (a, b) \in A. \end{aligned}$$

It is thus a neighbourhood of (x, y) which lies within A. Using the previous problem, we conclude that A is open in $X \times X$.

3. Suppose X is a Hausdorff space that has finitely many elements. Show that every subset of X is open in X. Hint: Use Theorem 2.14.

If A is a subset of X, then its complement is a finite subset of a Hausdorff space X, so it is closed in X. Since the complement of A is closed in X, we conclude that A is open in X.

4. Suppose $f\colon X \to Y$ is both continuous and injective. Suppose also that Y is Hausdorff. Show that X must be Hausdorff as well.

Suppose x, y are distinct points in X. Then $f(x), f(y)$ are distinct points in Y by injectivity. Since Y is Hausdorff, we can always find sets U, V which are open in Y with

$$f(x) \in U, \qquad f(y) \in V, \qquad U \cap V = \varnothing.$$

It follows by continuity that $f^{-1}(U), f^{-1}(V)$ are open in X with

$$x \in f^{-1}(U), \qquad y \in f^{-1}(V), \qquad f^{-1}(U) \cap f^{-1}(V) = \varnothing.$$

This shows that the space X is Hausdorff as well.

Homework 7. Solutions

1. Show that the unit circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Hint: Use polar coordinates and Theorem 2.15.

In terms of polar coordinates, the points on the unit circle are the points that have the form $(\cos\theta, \sin\theta)$. Consider the function

$$f\colon [0, 2\pi) \to \mathbb{R}^2, \qquad f(\theta) = (\cos\theta, \sin\theta).$$

Each component of f is continuous, so f itself is continuous. Since the domain of f is an interval, it is connected, so the image of f is connected as well. In other words, the unit circle C is connected.

2. Let $U = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$. How many connected components does this set have? Hint: One of the components is $(0,\infty) \times (0,\infty)$.

To say that $xy \neq 0$ is to say that each of x, y is either positive or negative. Thus, U can be expressed as the union of the sets

$$U_1 = (0,\infty) \times (0,\infty), \qquad U_2 = (0,\infty) \times (-\infty,0),$$
$$U_3 = (-\infty,0) \times (0,\infty), \qquad U_4 = (-\infty,0) \times (-\infty,0).$$

Each of these sets is a product of intervals and thus connected. It remains to show that they are actually connected components.

Let A be a connected subset of U and let $p_i \colon A \to \mathbb{R}$ denote the projection on the ith variable. By continuity, each $p_i(A)$ is an interval that does not contain 0, so it is contained in either $(-\infty,0)$ or $(0,\infty)$. Thus, A itself is contained in one of the sets U_j.

Homework 7. Solutions

3. Show that there is no surjective continuous function $f\colon A \to B$, if

$$A = (0,3) \cup (3,6), \qquad B = (0,1) \cup (1,2) \cup (2,3).$$

Hint: Look at the restriction of f on each subinterval. You will need to use Theorem 2.11 and also some parts of Theorem 2.15.

Note that $A = A_1 \cup A_2$ is the union of two disjoint open intervals and $B = B_1 \cup B_2 \cup B_3$ is the union of three disjoint open intervals. Since $f\colon A \to B$ is continuous, each restriction $f\colon A_i \to B$ must be continuous, so each $f(A_i)$ must be connected.

Since $f(A_i)$ is a connected subset of $B_1 \cup B_2 \cup B_3$, it lies within either B_1 or $B_2 \cup B_3$. If it actually lies in $B_2 \cup B_3$, then it lies within either B_2 or B_3. Thus, each $f(A_i)$ is contained in a single B_j and the function f is not surjective, contrary to assumption.

Homework 7. Solutions

4. Let (X, T) be a topological space and suppose $A_1, A_2, \ldots, A_n$ are connected subsets of X such that $A_k \cap A_{k+1}$ is nonempty for each k. Show that the union of these sets is connected. Hint: Use induction.

When $n = 1$, the union is equal to A_1 and this set is connected by assumption. Suppose that the result holds for n sets and consider the union of $n + 1$ sets. This union has the form

$$U = A_1 \cup \cdots \cup A_n \cup A_{n+1} = B \cup A_{n+1},$$

where B is connected by the induction hypothesis. Since A_{n+1} has a point in common with A_n, it has a point in common with B. In particular, the union $B \cup A_{n+1}$ is connected and the result follows.

Homework 8. Solutions

1. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^4 \leq 4\}$ is compact.

First of all, the set A is bounded because its points satisfy

$$\begin{aligned} x^2 \leq x^2 + 4y^4 \leq 4 \quad &\Longrightarrow \quad |x| \leq 2, \\ 4y^4 \leq x^2 + 4y^4 \leq 4 \quad &\Longrightarrow \quad |y| \leq 1. \end{aligned}$$

To show that A is also closed, we consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 + 4y^4.$$

Since f is continuous and $(-\infty, 4]$ is closed in $\mathbb{R}$, its inverse image is closed in $\mathbb{R}^2$. This means that A is closed in $\mathbb{R}^2$. Since A is both bounded and closed in $\mathbb{R}^2$, we conclude that A is compact.

2. Show that the set B is a compact subset of $\mathbb{R}^2$ when

$$B = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ \ y \geq 0,\ \ x + y \leq 1\}.$$

First of all, the set B is bounded because its points satisfy

$$0 \leq x \leq x + y \leq 1, \qquad 0 \leq y \leq x + y \leq 1.$$

To show that B is also closed, we consider the functions

$$f_1(x, y) = x, \qquad f_2(x, y) = y, \qquad f_3(x, y) = x + y.$$

These are all continuous functions and it is easy to see that

$$B = f_1^{-1}([0, \infty)) \cap f_2^{-1}([0, \infty)) \cap f_3^{-1}((-\infty, 1]).$$

In particular, B is closed in $\mathbb{R}^2$ and also bounded, so it is compact.

Homework 8. Solutions

3. Suppose A, B are compact subsets of a Hausdorff space X. Show that $A \cap B$ is compact. Hint: Use the first two parts of Theorem 2.19.

Since A is a compact subset of a Hausdorff space X, it is actually closed in X. This implies that $A \cap B$ is closed in B. Being a closed subset of a compact space, $A \cap B$ must also be compact.

4. Let C_n be a sequence of nonempty, closed subsets of a compact space X such that $C_n \supset C_{n+1}$ for each n. Show that the intersection of these sets is nonempty. Hint: One has $\bigcup (X - C_i) = X - \bigcap C_i$.

Suppose the intersection is empty. Then we actually have

$$X = X - \bigcap_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} (X - C_i),$$

so the sets $X - C_i$ form an open cover of X. Since X is compact, it is covered by finitely many sets, say the first k. This gives

$$X = \bigcup_{i=1}^{k} (X - C_i) = X - \bigcap_{i=1}^{k} C_i = X - C_k,$$

so the set C_k must be empty, contrary to assumption.