Homework 1. Solutions

1. Show that the discrete metric satisfies the properties of a metric.

The discrete metric is defined by the formula

$$d(x, y) = \left\{ \begin{array}{ll} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{array} \right\}.$$

It is clearly symmetric and non-negative with $d(x, y) = 0$ if and only if $x = y$. It remains to establish the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y).$$

If $x = y$, then the left hand side is zero and the inequality certainly holds. If $x \neq y$, then the left hand side is equal to 1. Since $x \neq y$, we must have either $z \neq x$ or else $z \neq y$. Thus, the right hand side is at least 1 and the triangle inequality holds in any case.

2. Compute the distances $d_1(f,g)$ and $d_\infty(f,g)$ when $f,g \in C[0,1]$ are the functions defined by $f(x)=x^2$ and $g(x)=x^3$.

Since $x^2 \geq x^3$ for all $x \in [0,1]$, the first distance is given by

$$d_1(f,g) = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

To compute the second distance, we need to find the maximum of

$$h(x) = x^2 - x^3, \qquad 0 \leq x \leq 1.$$

Since $h'(x) = 2x - 3x^2 = x(2-3x)$, it easily follows that

$$d_\infty(f,g) = h(2/3) = 4/9 - 8/27 = 4/27.$$

3. Sketch the open ball $B((0,0),1)$ in the metric space $(\mathbb{R}^2, d_\infty)$.

The open ball $B((0,0),1)$ contains the points (x,y) that satisfy

$$d_\infty((x,y),(0,0)) = \max\{|x|,|y|\} < 1.$$

Now, the maximum of two numbers is smaller than 1 if and only if the two numbers are both smaller than 1. This gives the condition

$$|x| < 1 \quad \text{and} \quad |y| < 1.$$

Thus, the open ball $B((0,0),1)$ is the interior of the square whose vertices are located at the points $(\pm 1, \pm 1)$.

4. Let $A = \{x \in \mathbb{R} : x > 0\}$. Is this set bounded in $(\mathbb{R}, d)$ when d is the usual metric? Is it bounded when d is the discrete metric?

This set is not bounded with respect to the usual metric. If it were bounded, then we would have

$$A \subset (x - r, x + r)$$

for some $x \in \mathbb{R}$ and some $r > 0$. This is not the case because

$$|x| + r \in A, \qquad |x| + r \notin (x - r, x + r).$$

To show that A is bounded with respect to the discrete metric, we note that A is contained in $B(x, 2) = \mathbb{R}$ for any $x \in \mathbb{R}$.

5. Consider a metric space (X, d) whose metric d is discrete. Show that every subset $A \subset X$ is open in X.

Let $x \in A$ and consider the open ball $B(x, 1)$. Since d is discrete, this open ball is equal to $\{x\}$, so it is contained entirely within A.

Homework 2. Solutions

1. Let (X, d) be a metric space. Given a point $x \in X$ and a real number $r > 0$, show that $U = \{y \in X : d(x, y) > r\}$ is open in X.

Let $y \in U$. Then $\varepsilon = d(x, y) - r$ is positive and we have

$$\begin{aligned} z \in B(y, \varepsilon) \quad &\implies \quad d(y, z) < \varepsilon \\ &\implies \quad r + d(y, z) < d(x, y) \leq d(x, z) + d(z, y) \\ &\implies \quad r < d(x, z) \\ &\implies \quad z \in U. \end{aligned}$$

This shows that $B(y, \varepsilon) \subset U$ and that the set U is open.

Homework 2. Solutions

2. Show that each of the following sets is closed in $\mathbb{R}$.

$$A = [0, \infty), \qquad B = \mathbb{Z}, \qquad C = \{x \in \mathbb{R} : \sin x \leq 0\}.$$

The complements of the given sets can be expressed in the form

$$\mathbb{R} - A = (-\infty, 0) = \bigcup_{n \in \mathbb{N}} (-n, 0),$$

$$\mathbb{R} - B = \bigcup_{x \in \mathbb{Z}} (x, x + 1),$$

$$\mathbb{R} - C = \{x \in \mathbb{R} : \sin x > 0\} = \bigcup_{k \in \mathbb{Z}} (2k\pi, 2k\pi + \pi).$$

These are all unions of open intervals, so they are all open in $\mathbb{R}$. Thus, the given sets A, B, C are all closed in $\mathbb{R}$.

Homework 2. Solutions

3. Find a collection of closed subsets of $\mathbb{R}$ whose union is not closed.

Needless to say, there are several examples. One typical example is

$$A_n = [1/n, 2] \quad \Longrightarrow \quad \bigcup_{n\in\mathbb{N}} A_n = (0, 2].$$

More generally, let $\{x_n\}$ be any strictly decreasing sequence of real numbers whose limit x is finite. Then it is easy to see that

$$A_n = [x_n, y] \quad \Longrightarrow \quad \bigcup_{n\in\mathbb{N}} A_n = (x, y].$$

4. Let (X, d_X) and (Y, d_Y) be metric spaces. Assuming that d_X is discrete, show that any function $f\colon X \to Y$ is continuous.

Let $x \in X$ and $\varepsilon > 0$ be given. Then we have

$$\begin{aligned} y \in B(x,1) \quad &\Longrightarrow \quad y = x \\ &\Longrightarrow \quad f(y) = f(x) \\ &\Longrightarrow \quad f(y) \in B(f(x), \varepsilon). \end{aligned}$$

This shows that f is continuous at the point x.

Homework 3. Solutions

1. Consider the sequence of functions defined by $f_n(x) = \frac{x}{1+n^2x^2}$ for all $x \geq 0$. Show that this sequence converges uniformly on $[0, \infty)$.

It is clear that $f_n(x)$ converges pointwise to the zero function. To show that it converges uniformly, we compute the supremum of

$$g_n(x) = |f_n(x) - 0| = \frac{x}{1 + n^2x^2}$$

on the interval $[0, \infty)$. Using the quotient rule, we get

$$g_n'(x) = \frac{1 + n^2x^2 - 2n^2x^2}{(1 + n^2x^2)^2} = \frac{1 - n^2x^2}{(1 + n^2x^2)^2}$$

and so g_n attains its maximum at the point $x = 1/n$. In particular, the supremum is $g_n(1/n) = 1/(2n)$ and it goes to zero as $n \to \infty$.

2. Let (X, d) be a metric space, let $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ uniformly on X and let x_n be a sequence of points of X with $x_n \to x$. Show that $f_n(x_n) \to f(x)$.

Let $\varepsilon > 0$ be given. Then there exists an integer N_1 such that

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{for all } n \geq N_1 \text{ and all } x \in X.$$

Since $f_n \to f$ uniformly, the limit f is continuous. Since $x_n \to x$, we must also have $f(x_n) \to f(x)$. Pick an integer N_2 such that

$$|f(x_n) - f(x)| < \varepsilon/2 \quad \text{for all } n \geq N_2.$$

Given any integer $n \geq \max\{N_1, N_2\}$, we must then have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.$$

3. Let $f: X \to Y$ be a function between metric spaces and let x_n be a Cauchy sequence in X. Show that $f(x_n)$ must also be Cauchy, if f is Lipschitz continuous. Is the same true, if f is merely continuous?

Suppose that f is Lipschitz continuous with constant $L > 0$. Given any $\varepsilon > 0$, we can then find an integer N such that

$$d(x_m, x_n) < \varepsilon/L \quad \text{for all } m, n \geq N.$$

Since f is Lipschitz continuous, this also implies that

$$d(f(x_m), f(x_n)) \leq L \cdot d(x_m, x_n) < \varepsilon \quad \text{for all } m, n \geq N$$

and so $f(x_n)$ is Cauchy. When f is merely continuous, the result is not true. For instance, $x_n = 1/n$ is Cauchy in $\mathbb{R}$ and $f(x) = 1/x$ is continuous, but $f(x_n) = n$ is certainly not Cauchy.

4. Show that (X, d) is complete, if the metric d is discrete.

Suppose that x_n is a Cauchy sequence in X. Then there exists an integer N such that

$$d(x_m, x_n) < 1 \quad \text{for all } m, n \geq N.$$

Since the metric d is discrete, this actually means that

$$x_m = x_n \quad \text{for all } m, n \geq N.$$

Given any $\varepsilon > 0$, we must then have

$$d(x_n, x_N) = d(x_N, x_N) < \varepsilon \quad \text{for all } n \geq N.$$

In other words, we must have $x_n \to x_N$ as $n \to \infty$.

Homework 4. Solutions

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $A = \{x \in \mathbb{R} : f(x) \geq 0\}$. Show that A is closed in $\mathbb{R}$ and conclude that A is complete.

The set $U = (-\infty, 0)$ is open in $\mathbb{R}$ because it can be written as

$$U = (-\infty, 0) = \bigcup_{n \in \mathbb{N}} (-n, 0)$$

and this is a union of open intervals. Since f is continuous,

$$f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\} = \{x \in \mathbb{R} : f(x) < 0\}$$

is then open in $\mathbb{R}$. Thus, the complement of $f^{-1}(U)$ is closed and this means that A is closed. Since $\mathbb{R}$ is complete and A is a closed subset of $\mathbb{R}$, we conclude that A is complete.

2. Suppose $f\colon [a,b] \to [a,b]$ is a differentiable function such that

$$L = \sup_{a \leq x \leq b} |f'(x)|$$

satisfies $L < 1$. Show that f has a unique fixed point in $[a,b]$.

Let $x, y \in [a,b]$. Using the mean value theorem, one finds that

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq L \cdot |x - y|$$

for some point c between x and y. Since $L < 1$ by assumption, this shows that f is a contraction on $[a,b]$. On the other hand, $[a,b]$ is a closed subset of $\mathbb{R}$ and thus complete. It follows by Banach's fixed point theorem that f has a unique fixed point in $[a,b]$.

3. Show that there is a unique real number x such that $\cos x = x$. Hint: Such a number must lie in $[-1, 1]$. Use the previous problem.

Since $f(x) = \cos x$ is between -1 and 1 for all x, every fixed point of f must lie in the interval $[-1, 1]$. Note that

$$L = \sup_{-1 \leq x \leq 1} |f'(x)| = \sup_{-1 \leq x \leq 1} |\sin x| = \sup_{0 \leq x \leq 1} \sin x = \sin 1$$

is strictly less than 1. In view of the previous problem, f must have a unique fixed point in $[-1, 1]$, so it has a unique fixed point in $\mathbb{R}$. Thus, there is a unique real number x such that $f(x) = x$.

4. Show that the set $\mathbb{Q}$ of all rational numbers is not complete.

Consider a sequence of rational numbers that converges to $\sqrt{2}$, say

$$x_1 = 1.4$$
$$x_2 = 1.41$$
$$x_3 = 1.414$$

and so on. This sequence is convergent in $\mathbb{R}$, so it is Cauchy, but it is not convergent in $\mathbb{Q}$ because its limit is irrational.