1. Show that the discrete metric satisfies the properties of a metric.

The discrete metric is defined by the formula

\[ d(x, y) = \begin{cases} 
1 & \text{if } x \neq y \\
0 & \text{if } x = y 
\end{cases} \]

It is clearly symmetric and non-negative with \( d(x, y) = 0 \) if and only if \( x = y \). It remains to establish the triangle inequality

\[ d(x, y) \leq d(x, z) + d(z, y). \]

If \( x = y \), then the left hand side is zero and the inequality certainly holds. If \( x \neq y \), then the left hand side is equal to 1. Since \( x \neq y \), we must have either \( z \neq x \) or else \( z \neq y \). Thus, the right hand side is at least 1 and the triangle inequality holds in any case.
2. Compute the distances $d_1(f, g)$ and $d_\infty(f, g)$ when $f, g \in C[0, 1]$ are the functions defined by $f(x) = x^2$ and $g(x) = x^3$.

Since $x^2 \geq x^3$ for all $x \in [0, 1]$, the first distance is given by

$$d_1(f, g) = \int_0^1 (x^2 - x^3) \, dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$ 

To compute the second distance, we need to find the maximum of

$$h(x) = x^2 - x^3, \quad 0 \leq x \leq 1.$$ 

Since $h'(x) = 2x - 3x^2 = x(2 - 3x)$, it easily follows that

$$d_\infty(f, g) = h(2/3) = 4/9 - 8/27 = 4/27.$$
3. Sketch the open ball $B((0, 0), 1)$ in the metric space $(\mathbb{R}^2, d_{\infty})$.

The open ball $B((0, 0), 1)$ contains the points $(x, y)$ that satisfy

$$d_{\infty}((x, y), (0, 0)) = \max\{|x|, |y|\} < 1.$$  

Now, the maximum of two numbers is smaller than 1 if and only if the two numbers are both smaller than 1. This gives the condition

$$|x| < 1 \quad \text{and} \quad |y| < 1.$$  

Thus, the open ball $B((0, 0), 1)$ is the interior of the square whose vertices are located at the points $(\pm1, \pm1)$. 
4. Let $A = \{x \in \mathbb{R} : x > 0\}$. Is this set bounded in $(\mathbb{R}, d)$ when $d$ is the usual metric? Is it bounded when $d$ is the discrete metric?

This set is not bounded with respect to the usual metric. If it were bounded, then we would have

$$A \subset (x - r, x + r)$$

for some $x \in \mathbb{R}$ and some $r > 0$. This is not the case because

$$|x| + r \in A, \quad |x| + r \notin (x - r, x + r).$$

To show that $A$ is bounded with respect to the discrete metric, we note that $A$ is contained in $B(x, 2) = \mathbb{R}$ for any $x \in \mathbb{R}$. 
5. Consider a metric space $(X, d)$ whose metric $d$ is discrete. Show that every subset $A \subset X$ is open in $X$.

Let $x \in A$ and consider the open ball $B(x, 1)$. Since $d$ is discrete, this open ball is equal to $\{x\}$, so it is contained entirely within $A$. 
1. Let \((X, d)\) be a metric space. Given a point \(x \in X\) and a real number \(r > 0\), show that \(U = \{y \in X : d(x, y) > r\}\) is open in \(X\).

Let \(y \in U\). Then \(\varepsilon = d(x, y) - r\) is positive and we have

\[
\begin{align*}
z \in B(y, \varepsilon) & \implies d(y, z) < \varepsilon \\
& \implies r + d(y, z) < d(x, y) \leq d(x, z) + d(z, y) \\
& \implies r < d(x, z) \\
& \implies z \in U.
\end{align*}
\]

This shows that \(B(y, \varepsilon) \subset U\) and that the set \(U\) is open.
2. Show that each of the following sets is closed in $\mathbb{R}$.

\[ A = [0, \infty), \quad B = \mathbb{Z}, \quad C = \{ x \in \mathbb{R} : \sin x \leq 0 \}. \]

The complements of the given sets can be expressed in the form

\[ \mathbb{R} - A = (-\infty, 0) = \bigcup_{n \in \mathbb{N}} (-n, 0), \]

\[ \mathbb{R} - B = \bigcup_{x \in \mathbb{Z}} (x, x + 1), \]

\[ \mathbb{R} - C = \{ x \in \mathbb{R} : \sin x > 0 \} = \bigcup_{k \in \mathbb{Z}} (2k\pi, 2k\pi + \pi). \]

These are all unions of open intervals, so they are all open in $\mathbb{R}$. Thus, the given sets $A, B, C$ are all closed in $\mathbb{R}$. 
3. Find a collection of closed subsets of $\mathbb{R}$ whose union is not closed.

Needless to say, there are several examples. One typical example is

$$A_n = [1/n, 2] \implies \bigcup_{n \in \mathbb{N}} A_n = (0, 2].$$

More generally, let $\{x_n\}$ be any strictly decreasing sequence of real numbers whose limit $x$ is finite. Then it is easy to see that

$$A_n = [x_n, y] \implies \bigcup_{n \in \mathbb{N}} A_n = (x, y].$$
4. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Assuming that $d_X$ is discrete, show that any function $f : X \to Y$ is continuous.

Let $x \in X$ and $\varepsilon > 0$ be given. Then we have

$$y \in B(x, 1) \implies y = x \implies f(y) = f(x) \implies f(y) \in B(f(x), \varepsilon).$$

This shows that $f$ is continuous at the point $x$. 
1. Consider the sequence of functions defined by \( f_n(x) = \frac{x}{1+n^2x^2} \) for all \( x \geq 0 \). Show that this sequence converges uniformly on \([0, \infty)\).

It is clear that \( f_n(x) \) converges pointwise to the zero function. To show that it converges uniformly, we compute the supremum of

\[
g_n(x) = |f_n(x) - 0| = \frac{x}{1+n^2x^2}
\]

on the interval \([0, \infty)\). Using the quotient rule, we get

\[
g'_n(x) = \frac{1 + n^2x^2 - 2n^2x^2}{(1 + n^2x^2)^2} = \frac{1 - n^2x^2}{(1 + n^2x^2)^2}
\]

and so \( g_n \) attains its maximum at the point \( x = 1/n \). In particular, the supremum is \( g_n(1/n) = 1/(2n) \) and it goes to zero as \( n \to \infty \).
2. Let \((X, d)\) be a metric space, let \(f_n: X \to \mathbb{R}\) be a sequence of continuous functions such that \(f_n \to f\) uniformly on \(X\) and let \(x_n\) be a sequence of points of \(X\) with \(x_n \to x\). Show that \(f_n(x_n) \to f(x)\).

Let \(\varepsilon > 0\) be given. Then there exists an integer \(N_1\) such that

\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1 \text{ and all } x \in X.
\]

Since \(f_n \to f\) uniformly, the limit \(f\) is continuous. Since \(x_n \to x\), we must also have \(f(x_n) \to f(x)\). Pick an integer \(N_2\) such that

\[
|f(x_n) - f(x)| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.
\]

Given any integer \(n \geq \max\{N_1, N_2\}\), we must then have

\[
|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.
\]
3. Let $f: X \to Y$ be a function between metric spaces and let $x_n$ be a Cauchy sequence in $X$. Show that $f(x_n)$ must also be Cauchy, if $f$ is Lipschitz continuous. Is the same true, if $f$ is merely continuous?

Suppose that $f$ is Lipschitz continuous with constant $L > 0$. Given any $\varepsilon > 0$, we can then find an integer $N$ such that

$$d(x_m, x_n) < \varepsilon / L \quad \text{for all } m, n \geq N.$$

Since $f$ is Lipschitz continuous, this also implies that

$$d(f(x_m), f(x_n)) \leq L \cdot d(x_m, x_n) < \varepsilon \quad \text{for all } m, n \geq N$$

and so $f(x_n)$ is Cauchy. When $f$ is merely continuous, the result is not true. For instance, $x_n = 1/n$ is Cauchy in $\mathbb{R}$ and $f(x) = 1/x$ is continuous, but $f(x_n) = n$ is certainly not Cauchy.
4. Show that \((X, d)\) is complete, if the metric \(d\) is discrete.

Suppose that \(x_n\) is a Cauchy sequence in \(X\). Then there exists an integer \(N\) such that

\[ d(x_m, x_n) < 1 \quad \text{for all } m, n \geq N. \]

Since the metric \(d\) is discrete, this actually means that

\[ x_m = x_n \quad \text{for all } m, n \geq N. \]

Given any \(\varepsilon > 0\), we must then have

\[ d(x_n, x_N) = d(x_N, x_N) < \varepsilon \quad \text{for all } n \geq N. \]

In other words, we must have \(x_n \to x_N\) as \(n \to \infty\).
1. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and let \( A = \{ x \in \mathbb{R} : f(x) \geq 0 \} \). Show that \( A \) is closed in \( \mathbb{R} \) and conclude that \( A \) is complete.

The set \( U = (-\infty, 0) \) is open in \( \mathbb{R} \) because it can be written as

\[
U = (-\infty, 0) = \bigcup_{n \in \mathbb{N}} (-n, 0)
\]

and this is a union of open intervals. Since \( f \) is continuous,

\[
f^{-1}(U) = \{ x \in \mathbb{R} : f(x) \in U \} = \{ x \in \mathbb{R} : f(x) < 0 \}
\]

is then open in \( \mathbb{R} \). Thus, the complement of \( f^{-1}(U) \) is closed and this means that \( A \) is closed. Since \( \mathbb{R} \) is complete and \( A \) is a closed subset of \( \mathbb{R} \), we conclude that \( A \) is complete.
2. Suppose $f : [a, b] \rightarrow [a, b]$ is a differentiable function such that

$$L = \sup_{a \leq x \leq b} |f'(x)|$$

satisfies $L < 1$. Show that $f$ has a unique fixed point in $[a, b]$.

Let $x, y \in [a, b]$. Using the mean value theorem, one finds that

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq L \cdot |x - y|$$

for some point $c$ between $x$ and $y$. Since $L < 1$ by assumption, this shows that $f$ is a contraction on $[a, b]$. On the other hand, $[a, b]$ is a closed subset of $\mathbb{R}$ and thus complete. It follows by Banach's fixed point theorem that $f$ has a unique fixed point in $[a, b]$. 
3. Show that there is a unique real number $x$ such that $\cos x = x$. Hint: Such a number must lie in $[-1, 1]$. Use the previous problem.

Since $f(x) = \cos x$ is between $-1$ and $1$ for all $x$, every fixed point of $f$ must lie in the interval $[-1, 1]$. Note that

$$L = \sup_{-1 \leq x \leq 1} |f'(x)| = \sup_{-1 \leq x \leq 1} |\sin x| = \sup_{0 \leq x \leq 1} \sin x = \sin 1$$

is strictly less than $1$. In view of the previous problem, $f$ must have a unique fixed point in $[-1, 1]$, so it has a unique fixed point in $\mathbb{R}$. Thus, there is a unique real number $x$ such that $f(x) = x$. 
4. Show that the set \( \mathbb{Q} \) of all rational numbers is not complete.

Consider a sequence of rational numbers that converges to \( \sqrt{2} \), say

\[
\begin{align*}
x_1 &= 1.4 \\
x_2 &= 1.41 \\
x_3 &= 1.414 \\
\end{align*}
\]

and so on. This sequence is convergent in \( \mathbb{R} \), so it is Cauchy, but it is not convergent in \( \mathbb{Q} \) because its limit is irrational.