Maths 212: Homework Solutions

95. Letting $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, we may use the triangle inequality to get

$$\begin{aligned} ||x+y||_p^p &= \sum_i |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &\leq \sum_i |x_i| \cdot |x_i + y_i|^{p-1} + \sum_i |y_i| \cdot |x_i + y_i|^{p-1}. \end{aligned}$$

Next, we apply Hölder's inequality. Since $q = p/(p-1)$ is such that $1/p + 1/q = 1$, we find

$$\begin{aligned} \sum_i |x_i| \cdot |x_i + y_i|^{p-1} &\leq \left(\sum_i |x_i|^p \right)^{1/p} \left(\sum_i |x_i + y_i|^{(p-1)q} \right)^{1/q} \\ &= ||x||_p \cdot ||x+y||_p^{p/q} \end{aligned}$$

because $(p-1)q = p$ by above. Note that a similar argument applies to give

$$\sum_i |y_i| \cdot |x_i + y_i|^{p-1} \leq ||y||_p \cdot ||x+y||_p^{p/q}$$

as well. Once we now combine the last three equations, we arrive at

$$||x+y||_p^p \leq \Big(||x||_p + ||y||_p \Big) \cdot ||x+y||_p^{p/q}.$$

In the case that $||x+y||_p$ is nonzero, this implies the desired inequality

$$||x+y||_p = ||x+y||_p^{p-p/q} \leq ||x||_p + ||y||_p.$$

When $||x+y||_p$ is zero, on the other hand, the desired inequality holds trivially.

96. Suppose that $x_n = (x_{n1}, x_{n2}, \ldots)$ is a Cauchy sequence in ℓ^p and let $\varepsilon > 0$. Then

$$|x_{mk} - x_{nk}|^p \leq \sum_{k=1}^\infty |x_{mk} - x_{nk}|^p = ||x_m - x_n||_p^p < \varepsilon^p$$

for large enough m, n. This means that x_{nk} is a Cauchy sequence in $\mathbb{R}$ for each k, so x_{nk} converges for each k. Suppose $x_{nk} \to a_k$ for each k and set $a = (a_1, a_2, \ldots)$. Since

$$\sum_{k=1}^\infty |x_{mk} - x_{nk}|^p < \varepsilon^p$$

for large enough m, n by above, we may let $n \to \infty$ to find that

$$\sum_{k=1}^\infty |x_{mk} - a_k|^p < \varepsilon^p \quad \Longrightarrow \quad ||x_m - a||_p < \varepsilon$$

for large enough m. This shows that our sequence x_m converges to a. Moreover,

$$||a||_p \leq ||a - x_m||_p + ||x_m||_p < \varepsilon + ||x_m||_p$$

for large enough m by above, so we also have $a \in \ell^p$, as needed.

97. Linearity is rather easy to establish. To see that T is also bounded, we note that

$$\sum_{i=1}^{\infty} a_i^2\, x_i^2 \leq ||a||_\infty^2 \sum_{i=1}^{\infty} x_i^2 \quad \Longrightarrow \quad ||Tx||_2 \leq ||a||_\infty ||x||_2.$$

This implies the inequality $||T|| \leq ||a||_\infty$. Let us now consider the sequence

$$y_k = (0, \ldots, 0, 1, 0, \ldots)$$

whose coordinates are all zero, except for the kth one, which is equal to 1. Noting that

$$Ty_k = (0, \ldots, 0, a_k, 0, \ldots) \quad \Longrightarrow \quad ||Ty_k||_2 = |a_k| = |a_k| \cdot ||y_k||_2,$$

we find that $||T|| \geq |a_k|$ for each k. Taking the supremum of both sides, we thus find

$$||T|| \geq \sup_k |a_k| = ||a||_\infty \quad \Longrightarrow \quad ||T|| = ||a||_\infty.$$

98. Pick an element $a = (a_1, a_2, \ldots)$ in ℓ^p and consider its truncated version

$$x_n = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in \ell_0.$$

Since the series $\sum |a_k|^p$ converges by assumption, it is clear that

$$||x_n - a||_p^p = \sum_{k=n+1}^{\infty} |a_k|^p \longrightarrow 0 \qquad \text{as } n \to \infty.$$

This shows that the sequence x_n of points in ℓ_0 converges to a, namely that $a \in \operatorname{Cl} \ell_0$.

99. Suppose that $x_n = (x_{n1}, x_{n2}, \ldots)$ is a Cauchy sequence in C_0 and let $\varepsilon > 0$. Then

$$|x_{mk} - x_{nk}| \leq \sup_k |x_{mk} - x_{nk}| = d_\infty(x_m, x_n) < \varepsilon$$

for large enough m, n. This shows that x_{nk} is a Cauchy sequence in $\mathbb{R}$ for each k, so x_{nk} converges for each k. Suppose $x_{nk} \to a_k$ for each k and set $a = (a_1, a_2, \ldots)$. Since

$$|x_{mk} - x_{nk}| \leq \sup_k |x_{mk} - x_{nk}| < \varepsilon$$

for large enough m, n by above, we may let $n \to \infty$ to find that

$$|x_{mk} - a_k| < \varepsilon \quad \Longrightarrow \quad d_\infty(x_m, a) = \sup_k |x_{mk} - a_k| \leq \varepsilon$$

for large enough m. This shows that our sequence x_m converges to a. Moreover,

$$|a_k| \leq |a_k - x_{mk}| + |x_{mk}| \leq d_\infty(a, x_m) + |x_{mk}|.$$

Letting $k \to \infty$ and then $m \to \infty$, one now finds that $a \in C_0$, as needed.

100a. Using the ℓ^1-norm for ℓ_0, one easily finds that

$$|Tx| \leq \sum_{n=1}^{\infty} |x_n| = ||x||_1 \quad \Longrightarrow \quad ||T|| \leq 1.$$

In particular, T is a bounded linear transformation, hence also continuous.

100b. Define a sequence x_n of elements of ℓ_0 by setting

$$x_1 = (1, 0, 0, \ldots), \qquad x_2 = (1, 1, 0, 0, \ldots), \qquad x_3 = (1, 1, 1, 0, 0, \ldots),$$

and so on. Then we have $Tx_n = n$ and also $||x_n||_2 = \sqrt{n}$ for each n, hence

$$\frac{|Tx_n|}{||x_n||_2} = \sqrt{n}$$

fails to be bounded. In particular, T fails to be continuous as well.

101. Linearity is rather easy to establish. To see that A is also bounded, we note that

$$|x_1 + \ldots + x_n| \leq n \cdot ||x||_\infty \quad \Longrightarrow \quad ||Ax||_\infty \leq ||x||_\infty \quad \Longrightarrow \quad ||A|| \leq 1.$$

Since the element $y = (1, 1, 1, \ldots)$ is such that

$$Ay = \left(1, \frac{1+1}{2}, \frac{1+1+1}{3}, \ldots\right) = (1, 1, 1, \ldots) = y,$$

we also have

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||} \geq \frac{||Ay||}{||y||} = 1.$$

102. Using Hölder's inequality, one easily finds that

$$|Tx| \leq \sum_{i=1}^{\infty} |a_i\, x_i| \leq ||a||_q \cdot ||x||_p$$

for each $x = (x_1, x_2, \ldots)$ in ℓ^p. This implies the inequality $||T|| \leq ||a||_q$.

- Next, we consider the sequence $y = (y_1, y_2, \ldots)$ which is obtained by setting

$$y_i = |a_i|^{q/p} \cdot \operatorname{sign} a_i$$

for each i. To see that y is an element of ℓ^p, we need only note that

$$||y||_p^p = \sum_{i=1}^{\infty} |y_i|^p = \sum_{i=1}^{\infty} |a_i|^q = ||a||_q^q$$

is finite by assumption. Using the fact that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Longrightarrow \quad \frac{q}{p} + 1 = q,$$

one now easily finds

$$Ty = \sum_{i=1}^{\infty} |a_i|^{q/p} \cdot a_i \operatorname{sign} a_i = \sum_{i=1}^{\infty} |a_i|^{q/p+1} = \sum_{i=1}^{\infty} |a_i|^q = ||a||_q^q.$$

Combining our computations above, we thus arrive at

$$Ty = ||a||_q^q = ||a||_q^{q/p+1} = ||y||_p \cdot ||a||_q \quad \Longrightarrow \quad ||T|| \geq ||a||_q.$$

103. Suppose X is contractible and let $a, b \in X$ be arbitrary. Since X is contractible, one can always find a homotopy $F\colon X \times I \to X$ between the identity function and the constant map $f(x) = b$. This also gives rise to a path $\gamma(t) = F(a, t)$ between a and b, since

$$\gamma(0) = F(a, 0) = a, \qquad \gamma(1) = F(a, 1) = b.$$

104. Let $F\colon X \times I \to Y$ be a homotopy between the f_i's and $G\colon Y \times I \to Z$ be a homotopy between the g_i's. Then the function $H(x, t) = G(F(x, t), t)$ is continuous with

$$H(x, 0) = G(F(x, 0), 0) = G(f_0(x), 0) = g_0(f_0(x)) = (g_0 \circ f_0)(x),$$
$$H(x, 1) = G(F(x, 1), 1) = G(f_1(x), 1) = g_1(f_1(x)) = (g_1 \circ f_1)(x).$$

In other words, $H\colon X \times I \to Z$ is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$.

105. Since $\gamma = \alpha * \beta$ is a path from x to z, the function $\widehat{\gamma}\colon \pi_1(X, x) \to \pi_1(X, z)$ is defined by

$$\widehat{\gamma}([f]) = [\overline{\gamma}] * [f] * [\gamma]$$

for each loop f around x. To simplify this expression, we note that

$$[e_x] = [\gamma] * [\overline{\gamma}] = [\alpha] * [\beta] * [\overline{\gamma}] \quad \Longrightarrow \quad [\overline{\alpha}] = [\beta] * [\overline{\gamma}] \quad \Longrightarrow \quad [\overline{\beta}] * [\overline{\alpha}] = [\overline{\gamma}].$$

Once we now combine the last two equations, we find that

$$\widehat{\gamma}([f]) = [\overline{\beta}] * [\overline{\alpha}] * [f] * [\alpha] * [\beta] = [\overline{\beta}] * \widehat{\alpha}([f]) * [\beta] = \widehat{\beta}(\widehat{\alpha}([f])).$$

106. Suppose that $\gamma_0, \gamma_1\colon I \to X$ are two paths having the same endpoints and set

$$F(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s) \qquad \text{for all } s, t \in I.$$

Due to the convexity of X, this gives rise to a function $F\colon I \times I \to X$. Note that

$$F(s, 0) = \gamma_0(s), \qquad F(s, 1) = \gamma_1(s)$$

by the definition of F. Since γ_0 and γ_1 have the same endpoints, we also have

$$F(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s) = (1 - t)\gamma_0(s) + t\gamma_0(s) = \gamma_0(s)$$

for $s = 0, 1$. This means that $F\colon I \times I \to X$ is a path homotopy between γ_0 and γ_1.

107. According to the definition of concatenation of paths, one has

$$(f \circ (g * h))(t) = \left\{ \begin{array}{ll} f(g(2t)) & \text{if } 0 \leq t \leq 1/2 \\ f(h(2t-1)) & \text{if } 1/2 \leq t \leq 1 \end{array} \right\}$$

as well as

$$((f \circ g) * (f \circ h))(t) = \left\{ \begin{array}{ll} f(g(2t)) & \text{if } 0 \leq t \leq 1/2 \\ f(h(2t-1)) & \text{if } 1/2 \leq t \leq 1 \end{array} \right\}.$$

This proves the equality

$$f \circ (g * h) = (f \circ g) * (f \circ h) \quad \implies \quad f_*(g * h) = f_*(g) * f_*(h)$$

and it also implies that f_* is a group homomorphism.

108. Let $r\colon X \to A$ be the retraction and $i\colon A \to X$ be inclusion. Then the composition

$$A \xrightarrow{\ i\ } X \xrightarrow{\ r\ } A$$

is the identity map, so the composition

$$\pi_1(A, x_0) \xrightarrow{\ i_*\ } \pi_1(X, x_0) \xrightarrow{\ r_*\ } \pi_1(A, x_0)$$

is the identity map as well. This actually implies that r_* is surjective because

$$[\alpha] \in \pi_1(A, x_0) \quad \implies \quad [\alpha] = r_*(i_*([\alpha])).$$

109. First, suppose that $\pi_1(X, x_0)$ is abelian and let α, β be paths from x_0 to x_1. Since $\alpha * \overline{\beta}$ is a loop around x_0, we must then have

$$[f] * [\alpha] * [\overline{\beta}] = [f] * [\alpha * \overline{\beta}] = [\alpha * \overline{\beta}] * [f] = [\alpha] * [\overline{\beta}] * [f]$$

for each loop f around x_0. In view of the definition of $\widehat{\alpha}$, we must thus have

$$\widehat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha] * [\overline{\beta}] * [\beta] = [\overline{\alpha}] * [\alpha] * [\overline{\beta}] * [f] * [\beta] = \widehat{\beta}([f])$$

for each loop f around x_0.

- Now, suppose that $\widehat{\alpha} = \widehat{\beta}$ for all paths α, β from x_0 to x_1 and let f, h be loops around x_0. Since X is path connected, we can always find a path α from x_0 to x_1. Then $\beta = h * \alpha$ is also a path from x_0 to x_1, and we have

$$\beta = h * \alpha \quad \implies \quad \widehat{\beta} = \widehat{\alpha} \circ \widehat{h}$$

by Problem 105. Since $\widehat{\alpha} = \widehat{\beta}$ is an isomorphism, this actually implies that

$$\widehat{\alpha}([f]) = \widehat{\beta}([f]) = \widehat{\alpha}(\widehat{h}([f])) \quad \implies \quad [f] = \widehat{h}([f]).$$

In particular, it implies that

$$[f] = [\overline{h}] * [f] * [h] \quad \implies \quad [h] * [f] = [f] * [h].$$

110. Suppose that U is an open subset of Y and write its inverse image in the form

$$p^{-1}(U) = X \times U = \bigcup_{x \in X} \{x\} \times U.$$

Since X has the discrete topology, each of the sets $V_x = \{x\} \times U$ is open in $X \times Y$. Also, the restriction $p_x \colon V_x \to U$ is clearly a homeomorphism with inverse $q_x(y) = (x, y)$. Since the projection p is continuous and surjective, it is thus a covering map as well.

111. Suppose X is Hausdorff. Let $p \colon Y \to X$ be a covering map and $y_1 \neq y_2$ be elements of Y.

- Assuming that $p(y_1) \neq p(y_2)$, these two elements have disjoint neighbourhoods in X; the inverse images of these neighbourhoods are then disjoint neighbourhoods of the y_i's in Y.
- Assuming that $p(y_1) = p(y_2)$, this element has a neighbourhood U which is evenly covered by p; write its inverse image $p^{-1}(U) = \cup V_\alpha$ as a disjoint union of open sets. Were the y_i's lying in the same V_α, the restriction $p_\alpha \colon V_\alpha \to U$ would fail to be injective, which is not the case. Thus, there are two disjoint V_α's containing y_1 and y_2, as needed.

112. Letting $i \colon S^1 \to B^2$ be the inclusion map and $r \colon B^2 \to S^1$ a retraction, one has maps

$$S^1 \xrightarrow{\ i\ } B^2 \xrightarrow{\ r\ } S^1$$

whose composition is the identity. The composition of the induced homomorphisms

$$\pi_1(S^1, y_0) \xrightarrow{\ i_*\ } \pi_1(B^2, y_0) \xrightarrow{\ r_*\ } \pi_1(S^1, y_0)$$

must then be the identity map as well. Since the fundamental group $\pi_1(B^2, y_0)$ consists of one element only, the same is true for the image of r_*. This makes the image of $r_* \circ i_*$ consist of one element only, contrary to the fact that the image is $\pi_1(S^1, y_0) = \mathbb{Z}$.

113. Letting g be the inverse of f, we have maps

$$(X, x_0) \xrightarrow{\ f\ } (Y, y_0) \xrightarrow{\ g\ } (X, x_0) \xrightarrow{\ f\ } (Y, y_0)$$

such that both $f \circ g$ and $g \circ f$ are identity maps. Thus, the induced homomorphisms

$$\pi_1(X, x_0) \xrightarrow{\ f_*\ } \pi_1(Y, y_0) \xrightarrow{\ g_*\ } \pi_1(X, x_0) \xrightarrow{\ f_*\ } \pi_1(Y, y_0)$$

are such that both $f_* \circ g_*$ and $g_* \circ f_*$ are identity maps. This makes f_* an isomorphism.

114. Let us denote by A the set of all $x \in X$ for which $p^{-1}(x)$ has exactly n elements.

- To show that A is open in X, suppose $x \in A$ and let U be a neighbourhood of x which is evenly covered by p. Then $p^{-1}(U)$ can be written as a disjoint union

$$p^{-1}(U) = \bigcup_\alpha V_\alpha$$

such that $p_\alpha \colon V_\alpha \to U$ is a homeomorphism. Since $p^{-1}(x)$ contains exactly n elements, the union above contains exactly n sets. In particular, the inverse image of each $y \in U$ contains exactly n elements, and this forces U to lie entirely within A.

- Using the exact same argument, one can also show that the complement of A is open as well. Namely, each point $x \in X - A$ has a neighbourhood U such that the union

$$p^{-1}(U) = \bigcup_{\alpha} V_\alpha$$

does not contain n sets, and this forces U to lie entirely within $X - A$.

- Since X is connected, the only subsets of X which are both open and closed in X are the trivial ones, namely $\varnothing$ and X. Moreover, we already know that A is both open and closed in X. Since A is nonempty by assumption, this actually implies that $A = X$.

115. Let $x_0 \in X$ be arbitrary. Since Y is path connected by assumption, the natural map

$$\Phi \colon \pi_1(X, x_0) \to p^{-1}(x_0)$$

is surjective. Note that $\pi_1(X, x_0)$ contains only one element since X is simply connected. This implies that $p^{-1}(x_0)$ contains only one element as well. In particular, p is bijective, hence also a homeomorphism.

116. Suppose $A \subset X$ is a retract of a contractible space X and let $r \colon X \to A$ be a retraction. We have to show that any two continuous functions $f_0, f_1 \colon Y \to A$ are homotopic to one another. Thus, we need only show that any continuous function $f_1 \colon Y \to A$ is homotopic to the constant map $f_2(y) = a$.

- Let $i \colon A \to X$ denote the inclusion and consider the composition $i \circ f_1 \colon Y \to X$. Since X is contractible, we can always find a homotopy $F \colon Y \times [0, 1] \to X$ between $i \circ f_1$ and the constant map $f_2(y) = a$. Now, define the function $G \colon Y \times [0, 1] \to A$ by the formula

$$G(y, t) = r(F(y, t)).$$

Then G is a homotopy between f_1 and f_2 because G is continuous with

$$G(y, 0) = r(F(y, 0)) = r(f_1(y)) = f_1(y),$$
$$G(y, 1) = r(F(y, 1)) = r(a) = a = f_2(y).$$

117a. The unit disc B^2 with the origin removed has S^1 as a strong deformation retract. Thus, its fundamental group is isomorphic to $\pi_1(S^1) = \mathbb{Z}$.

117b. The complement of the z-axis in $\mathbb{R}^3$ has $S^1 \times \{0\}$ as a strong deformation retract. Since this space is homeomorphic to S^1, its fundamental group is then $\pi_1(S^1) = \mathbb{Z}$.

117c. The complement of the unit ball in $\mathbb{R}^3$ has S^2 as a strong deformation retract, so its fundamental group is $\pi_1(S^2) = 0$.

117d. The unit sphere S^2 with two points removed is homeomorphic to the punctured plane. Since the latter space has S^1 as a strong deformation retract, its fundamental group is $\mathbb{Z}$.

118. Let $r\colon B^2 \to X$ be the retraction and $i\colon X \to B^2$ be inclusion. Then the composition

$$B^2 \xrightarrow{\quad r \quad} X \xrightarrow{\quad f \quad} X \xrightarrow{\quad i \quad} B^2$$

must have a fixed point by Brouwer's theorem. Let $x_0 \in B^2$ be such, and note that

$$x_0 = i(f(r(x_0))) \quad \Longrightarrow \quad x_0 = f(r(x_0)) \in X.$$

Since the retraction r satisfies $r(x) = x$ for all $x \in X$, this actually implies that

$$x_0 = f(r(x_0)) = f(x_0).$$

119. Consider the functions

$$f(x,y) = (x^2 + x)\cos t - y^2 \sin t, \qquad g(x,y) = (x^2 - 1)\sin t + y^3 \cos t$$

and

$$h(x,y) = \left(\frac{f(x,y)}{5}, \frac{g(x,y)}{5} \right).$$

Then the given system of equations has a solution if and only if h has a fixed point. In view of Brouwer's theorem, it thus suffices to show that h maps B^2 to itself.

- Suppose now that (x,y) is an arbitrary point in B^2. Then

$$x^2 + y^2 \leq 1 \quad \Longrightarrow \quad x^2 \leq 1, \quad y^2 \leq 1$$

and this implies

$$|f(x,y)| \leq |x^2 \cos t| + |x \cos t| + |y^2 \sin t| \leq 3$$

as well as

$$|g(x,y)| \leq |x^2 \sin t| + |\sin t| + |y^3 \cos t| \leq 3.$$

In particular, it implies

$$\frac{f(x,y)^2}{25} + \frac{g(x,y)^2}{25} \leq \frac{9}{25} + \frac{9}{25} \leq 1 \quad \Longrightarrow \quad h(x,y) \in B^2.$$

120. Since the given loop has $\widetilde{\gamma}_0(t) = 4\pi i t$ as its lifting, its winding number is

$$w(\gamma_0) = \frac{\widetilde{\gamma}_0(1) - \widetilde{\gamma}_0(0)}{2\pi i} = \frac{4\pi i - 0}{2\pi i} = 2.$$

121. The given loop is homotopic to that of the previous exercise via the homotopy

$$F(s,t) = (1+t)\, e^{4\pi i s}.$$

Since homotopic loops have the same winding number, we deduce that $w(\gamma_1) = 2$ as well.

122. We already proved this in class.

123. Let X_1 and X_2 denote the two spheres and let $p \in X_1 \cap X_2$ denote the point in common. Pick any point $q_1 \in X_1 - \{p\}$ and any point $q_2 \in X_2 - \{p\}$. Then

$$X = (X - \{q_1\}) \cup (X - \{q_2\})$$

is the union of two simply connected sets whose intersection is path connected. In view of van Kampen's theorem, this also implies that X is simply connected.

124. The function $f(x) = -x$ is continuous on S^1. However, it has no fixed point because

$$f(x) = x \quad \Longrightarrow \quad -x = x \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad x \notin S^1.$$

125. Define a function $f\colon X \to [0, \infty)$ by the formula $f(x, y) = x^2 + y^2$. Since f attains the same value on equivalent points, it gives rise to a continuous function $\overline{f}\colon \overline{X} \to [0, \infty)$. To see that $\overline{f}$ is actually a homeomorphism, consider the composition

$$[0, \infty) \xrightarrow{\ g\ } X \xrightarrow{\ p\ } \overline{X},$$

where g is defined by $g(r) = (\sqrt{r}, 0)$ and p is the map which sends each element of X to its equivalence class. Then $p \circ g$ is the inverse of $\overline{f}$ because

$$\overline{f}(p(g(r))) = \overline{f}([g(r)]) = f(g(r)) = r.$$

Moreover, $p \circ g$ is the composition of continuous functions, hence also continuous.

126. Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are quotient maps. Then each of f, g is surjective, so their composition $g \circ f$ is surjective as well. Moreover, we have

$$U \text{ is open in } Z \iff g^{-1}(U) \text{ is open in } Y \iff f^{-1}(g^{-1}(U)) \text{ is open in } X.$$

Since this implies that

$$U \text{ is open in } Z \iff (g \circ f)^{-1}(U) \text{ is open in } X,$$

we may conclude that the composition $g \circ f$ is a quotient map itself.

127. The quotient space is homeomorphic to S^2. An easy way to visualize the identification is by thinking of soap bubbles. If you take a circular layer of soap and blow into it, then the interior of the circle will move forward and its boundary will eventually collapse into a single point; this also gives rise to a bubble.

128. Suppose we can find a continuous function $f\colon X \to S^2$ which is constant throughout the boundary. Then f would give rise to a continuous function $\overline{f}\colon \overline{X} \to S^2$ from a compact space to a Hausdorff space, and this would actually be a homeomorphism by Problem 43.

- To find a continuous function $f\colon B^2 \to S^2$ that is constant throughout the boundary, we express the domain B^2 in terms of polar coordinates

$$B^2 = \{(r\cos\theta, r\sin\theta) \,:\, 0 \le \theta < 2\pi, \quad 0 \le r \le 1\}$$

and the image S^2 in terms of spherical coordinates

$$S^2 = \{(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) \,:\, 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi\}.$$

In view of the last two equations, it is now easy to see that the function

$$f(r\cos\theta, r\sin\theta) = \big(\cos\theta\sin(\pi r), \sin\theta\sin(\pi r), \cos(\pi r)\big)$$

has the desired properties. Namely, this function maps the boundary $r = 1$ to the point

$$f(\cos\theta, \sin\theta) = (\cos\theta\sin\pi, \sin\theta\sin\pi, \cos\pi) = (0, 0, -1).$$

129. All positive numbers are equivalent to 1 and all negative numbers are equivalent to -1. The set of equivalence classes is thus easily found to be

$$\overline{X} = \{[0], [1], [-1]\}.$$

Let $p\colon X \to \overline{X}$ be the map that sends each real number x to its equivalence class $[x]$. By the definition of the quotient topology, the sets which are open in $\overline{X}$ are those whose inverse images are open in $X = \mathbb{R}$. In particular, the only open subsets of $\overline{X}$ are

$$\varnothing, \qquad \{[1]\}, \qquad \{[-1]\}, \qquad \{[1], [-1]\}, \qquad \overline{X}.$$

For instance, the inverse image of [1] is $(0, \infty)$, and so on. Since [0] and [1] fail to have disjoint neighbourhoods by above, we deduce that $\overline{X}$ is not Hausdorff.

130. Suppose that $r\colon X \to A$ is a retraction. Given any subset $U \subset A$, we then have

$$r^{-1}(U) \cap A = \{x \in A \,:\, r(x) \in U\} = \{x \in A \,:\, x \in U\} = U.$$

In particular, U is open in A if and only if $r^{-1}(U)$ is open in X.

131. Consider the function $f\colon X \to [-1, 1]$ defined by $f(x, y, z) = z$. Since f attains the same value on equivalent points, it gives rise to a continuous function $\overline{f}\colon \overline{X} \to [-1, 1]$. Noting that $\overline{X}$ is compact and $[-1, 1]$ is Hausdorff, we may then invoke Problem 43 to conclude that $\overline{f}$ is actually a homeomorphism.

132. We use the notation of Figure 1; see the last page. The 0-boundaries are generated by

$$\partial\alpha = \partial\gamma = x - y, \qquad \partial\beta = \partial\delta = 0$$

and this implies $B_0 = \mathbb{Z}[x - y]$. The 1-boundaries are generated by

$$\partial U = \gamma - \beta - \alpha, \qquad \partial L = \delta + \alpha - \gamma$$

and so $B_1 = \mathbb{Z}^2[\gamma - \beta - \alpha, \delta + \alpha - \gamma]$. On the other hand, $B_i = \{0\}$ for each $i \ge 2$.

- Next, we turn to the cycles. It is clear that $Z_0 = C_0 = \mathbb{Z}^2[x, y]$ because every 0-chain has zero boundary by definition. To find Z_1, we note that

$$\begin{aligned} k \cdot \partial\alpha + l \cdot \partial\beta + m \cdot \partial\gamma + n \cdot \partial\delta = 0 &\iff k(x-y) + m(x-y) = 0 \\ &\iff m = -k. \end{aligned}$$

This gives three degrees of freedom for the parameters k, l, m, n and it also implies

$$Z_1 = \{k\alpha + l\beta - k\gamma + n\delta \,:\, k, l, n \in \mathbb{Z}\} = \mathbb{Z}^3[\alpha - \gamma, \beta, \delta].$$

To find Z_2, we note that

$$\begin{aligned} m \cdot \partial U + n \cdot \partial L = 0 &\iff m(\gamma - \beta - \alpha) + n(\delta + \alpha - \gamma) = 0 \\ &\iff m = n = 0. \end{aligned}$$

This makes Z_2 trivial, so we actually have $Z_i = \{0\}$ for each $i \geq 2$.

- Finally, we turn to the homology groups. The first two of those are given by

$$\begin{aligned} H_0 &= \frac{\mathbb{Z}^2[x, y]}{\mathbb{Z}[x-y]} = \frac{\mathbb{Z}^2[x-y, y]}{\mathbb{Z}[x-y]} = \mathbb{Z}[y], \\ H_1 &= \frac{\mathbb{Z}^3[\alpha - \gamma, \beta, \delta]}{\mathbb{Z}^2[\gamma - \beta - \alpha, \delta + \alpha - \gamma]} = \frac{\mathbb{Z}^2[\gamma - \alpha, \beta]}{\mathbb{Z}[\gamma - \alpha - \beta]} = \mathbb{Z}[\beta]. \end{aligned}$$

Moreover, all higher homology groups are trivial by above.

133. We use the notation of Figure 1; see the last page. The 0-boundaries are generated by

$$\partial\alpha = y - x, \qquad \partial\beta = z - y, \qquad \partial\gamma = z - x = \partial\alpha + \partial\beta$$

and this implies $B_0 = \mathbb{Z}^2[y - x, z - y]$. The 1-boundaries are generated by

$$\partial A = \alpha + \beta - \gamma$$

and so $B_1 = \mathbb{Z}[\alpha + \beta - \gamma]$. On the other hand, $B_i = \{0\}$ for each $i \geq 2$.

- Next, we turn to the cycles. It is clear that $Z_0 = C_0 = \mathbb{Z}^3[x, y, z]$ because every 0-chain has zero boundary by definition. To find Z_1, we note that

$$\begin{aligned} k \cdot \partial\alpha + l \cdot \partial\beta + m \cdot \partial\gamma = 0 &\iff k(y-x) + l(z-y) + m(z-x) = 0 \\ &\iff k + m = k - l = l + m = 0 \\ &\iff l = k, \quad m = -k. \end{aligned}$$

This gives one degree of freedom for the parameters k, l, m and it also implies

$$Z_1 = \{k\alpha + k\beta - k\gamma \,:\, k \in \mathbb{Z}\} = \mathbb{Z}[\alpha + \beta - \gamma].$$

On the other hand, it is easy to see that $Z_i = \{0\}$ for each $i \geq 2$.

- Finally, we turn to the homology groups. The first two of those are given by

$$H_0 = \frac{\mathbb{Z}^3[x, y, z]}{\mathbb{Z}^2[y - x, z - y]} = \frac{\mathbb{Z}^2[x, y]}{\mathbb{Z}[y - x]} = \mathbb{Z}[x],$$
$$H_1 = \frac{\mathbb{Z}[\alpha + \beta - \gamma]}{\mathbb{Z}[\alpha + \beta - \gamma]} = \{0\}.$$

  In fact, all higher homology groups are trivial as well.

134. We use the notation of Figure 1; see the last page. The 0-boundaries are generated by

$$\partial\alpha = y - x, \qquad \partial\beta = x - y, \qquad \partial\gamma = 0, \qquad \partial\delta = y - x$$

and this implies $B_0 = \mathbb{Z}[y - x]$. The 1-boundaries are generated by

$$\partial U = \gamma - \beta - \alpha, \qquad \partial L = \delta - \alpha - \gamma$$

and so $B_1 = \mathbb{Z}^2[\gamma - \beta - \alpha, \delta - \alpha - \gamma]$. On the other hand, $B_i = \{0\}$ for each $i \geq 2$.

- Next, we turn to the cycles. It is clear that $Z_0 = C_0 = \mathbb{Z}^2[x, y]$ because every 0-chain has zero boundary by definition. To find Z_1, we note that

$$\begin{aligned} k \cdot \partial\alpha + l \cdot \partial\beta + m \cdot \partial\gamma + n \cdot \partial\delta = 0 &\iff k(y - x) + l(x - y) + n(y - x) = 0 \\ &\iff n = l - k. \end{aligned}$$

  This gives three degrees of freedom for the parameters k, l, m, n and it also implies

$$Z_1 = \{k\alpha + l\beta + m\gamma + (l - k)\delta \ : \ k, l, m \in \mathbb{Z}\} = \mathbb{Z}^3[\alpha - \delta, \beta + \delta, \gamma].$$

  On the other hand, it is easy to see that $Z_i = \{0\}$ for each $i \geq 2$.

- Finally, we turn to the homology groups. The first two of those are given by

$$H_0 = \frac{\mathbb{Z}^2[x, y]}{\mathbb{Z}[y - x]} = \frac{\mathbb{Z}^2[x, y - x]}{\mathbb{Z}[y - x]} = \mathbb{Z}[x],$$
$$H_1 = \frac{\mathbb{Z}^3[\alpha - \delta, \beta + \delta, \gamma]}{\mathbb{Z}^2[\gamma - \beta - \alpha, \delta - \alpha - \gamma]} = \frac{\mathbb{Z}^2[\alpha - \delta, \gamma]}{\mathbb{Z}[\delta - \alpha - \gamma]} = \mathbb{Z}[\gamma].$$

  Moreover, all higher homology groups are trivial by above.

135. We use the notation of Figure 2; see the last page. The 0-boundaries are generated by

$$\partial\alpha = \partial\beta = \partial\gamma = 0 \quad \Longrightarrow \quad B_0 = \{0\}.$$

The 1-boundaries are generated by

$$\partial U = \gamma - \beta - \alpha, \qquad \partial L = \alpha - \gamma + \beta = -\partial U$$

and so $B_1 = \mathbb{Z}[\gamma - \beta - \alpha]$. On the other hand, $B_i = \{0\}$ for each $i \geq 2$.

- Next, we turn to the cycles. We note that $Z_0 = C_0 = \mathbb{Z}[x]$ because every 0-chain has zero boundary by definition, while $Z_1 = C_1 = \mathbb{Z}^3[\alpha, \beta, \gamma]$ because every edge has zero boundary by above. To find Z_2, we note that

$$\begin{aligned} m \cdot \partial U + n \cdot \partial L = 0 &\iff m(\gamma - \beta - \alpha) + n(\alpha - \gamma + \beta) = 0 \\ &\iff n = m. \end{aligned}$$

  This gives one degree of freedom for the parameters m, n and it also implies

$$Z_2 = \{mU + mL \,:\, m \in \mathbb{Z}\} = \mathbb{Z}[U + L].$$

  On the other hand, it is clear that $Z_i = \{0\}$ for each $i \geq 3$.

- Finally, we turn to the homology groups. The first three of those are given by

$$\begin{aligned} H_0 &= \frac{\mathbb{Z}[x]}{\{0\}} = \mathbb{Z}[x], \\ H_1 &= \frac{\mathbb{Z}^3[\alpha, \beta, \gamma]}{\mathbb{Z}[\gamma - \beta - \alpha]} = \mathbb{Z}^2[\alpha, \beta], \\ H_2 &= \frac{\mathbb{Z}[U + L]}{\{0\}} = \mathbb{Z}[U + L]. \end{aligned}$$

  Moreover, all higher homology groups are trivial by above.

136. We use the notation of Figure 2; see the last page. The 0-boundaries are generated by

$$\partial\alpha = x - x = 0, \qquad \partial\beta = x - x = 0$$

and this implies that $B_0 = \{0\}$. In particular, it implies that $B_i = \{0\}$ for all i.

- When it comes to the cycles, it is easy to see that

$$Z_0 = C_0 = \mathbb{Z}[x], \qquad Z_1 = C_1 = \mathbb{Z}^2[\alpha, \beta], \qquad Z_2 = Z_3 = \cdots = \{0\}$$

  because both the vertex x and the edges α, β have zero boundary by above.

- Finally, we turn to the homology groups. The first two of those are given by

$$H_0 = \frac{\mathbb{Z}[x]}{\{0\}} = \mathbb{Z}[x], \qquad H_1 = \frac{\mathbb{Z}^2[\alpha, \beta]}{\{0\}} = \mathbb{Z}^2[\alpha, \beta]$$

  and all higher homology groups are trivial.

137. We use the notation of Figure 2; see the last page. The 0-boundaries are generated by

$$\partial\alpha = \partial\beta = \partial\gamma = 0$$

and this implies that $B_0 = \{0\}$. The 1-boundaries are generated by

$$\partial U = \gamma - \beta - \alpha, \qquad \partial L = \beta - \alpha - \gamma$$

and so $B_1 = \mathbb{Z}^2[\gamma - \beta - \alpha, \beta - \alpha - \gamma]$. On the other hand, $B_i = \{0\}$ for each $i \geq 2$.

- Next, we turn to the cycles. We note that $Z_0 = C_0 = \mathbb{Z}[x]$ because every 0-chain has zero boundary by definition, while $Z_1 = C_1 = \mathbb{Z}^3[\alpha, \beta, \gamma]$ because every edge has zero boundary by above. To find Z_2, we note that

$$\begin{aligned} m \cdot \partial U + n \cdot \partial L = 0 &\iff m(\gamma - \beta - \alpha) + n(\beta - \alpha - \gamma) = 0 \\ &\iff m + n = m - n = 0 \\ &\iff m = n = 0. \end{aligned}$$

  This also implies that $Z_i = \{0\}$ for any $i \geq 2$ whatsoever.

- Finally, we turn to the homology groups. The first two of those are given by

$$\begin{aligned} H_0 &= \frac{\mathbb{Z}[x]}{\{0\}} = \mathbb{Z}[x], \\ H_1 &= \frac{\mathbb{Z}^3[\alpha, \beta, \gamma]}{\mathbb{Z}^2[\gamma - \beta - \alpha, \beta - \alpha - \gamma]} = \frac{\mathbb{Z}^3[\alpha, \beta, \gamma - \beta - \alpha]}{\mathbb{Z}^2[\gamma - \beta - \alpha, 2\alpha]} = \frac{\mathbb{Z}[\alpha]}{\mathbb{Z}[2\alpha]} \times \mathbb{Z}[\beta]. \end{aligned}$$

  Moreover, all higher homology groups are trivial by above.

138. We use the notation of Figure 3; see the last page. The 0-boundaries are generated by

$$\partial\alpha = x - y, \qquad \partial\beta = x - y, \qquad \partial\gamma = 0$$

and this implies that $B_0 = \mathbb{Z}[x - y]$. The 1-boundaries are generated by

$$\partial U = \gamma + \beta - \alpha, \qquad \partial L = \beta - \alpha - \gamma$$

and so $B_1 = \mathbb{Z}^2[\gamma + \beta - \alpha, \beta - \alpha - \gamma]$. On the other hand, $B_i = \{0\}$ for each $i \geq 2$.

- Next, we turn to the cycles. We note that $Z_0 = C_0 = \mathbb{Z}^2[x, y]$ because every 0-chain has zero boundary by definition. To find Z_1, we note that

$$\begin{aligned} k \cdot \partial\alpha + l \cdot \partial\beta + m \cdot \partial\gamma = 0 &\iff k(x - y) + l(x - y) = 0 \\ &\iff l = -k. \end{aligned}$$

  This gives two degrees of freedom for the parameters k, l, m and it also implies

$$Z_1 = \{k\alpha - k\beta + m\gamma \ : \ k, m \in \mathbb{Z}\} = \mathbb{Z}^2[\alpha - \beta, \gamma].$$

  To find Z_2, we similarly note that

$$\begin{aligned} m \cdot \partial U + n \cdot \partial L = 0 &\iff m(\gamma + \beta - \alpha) + n(\beta - \alpha - \gamma) = 0 \\ &\iff m + n = m - n = 0 \\ &\iff m = n = 0. \end{aligned}$$

  This also implies that $Z_i = \{0\}$ for any $i \geq 2$ whatsoever.

- Finally, we turn to the homology groups. The first two of those are given by

$$H_0 = \frac{\mathbb{Z}^2[x,y]}{\mathbb{Z}[x-y]} = \frac{\mathbb{Z}^2[x-y,y]}{\mathbb{Z}[x-y]} = \mathbb{Z}[y],$$

$$H_1 = \frac{\mathbb{Z}^2[\alpha-\beta,\gamma]}{\mathbb{Z}^2[\gamma+\beta-\alpha,\beta-\alpha-\gamma]} = \frac{\mathbb{Z}^2[\alpha-\beta,\gamma+\beta-\alpha]}{\mathbb{Z}^2[\gamma+\beta-\alpha,2\beta-2\alpha]} = \frac{\mathbb{Z}[\alpha-\beta]}{\mathbb{Z}[2\alpha-2\beta]}\,.$$

  Moreover, all higher homology groups are trivial by above.

139. We use the notation of Figure 3; see the last page. Since $\partial x = \partial\alpha = 0$, we have $B_i = \{0\}$ for any $i \geq 0$ whatsoever. When it comes to the cycles, it is easy to see that

$$Z_0 = C_0 = \mathbb{Z}[x], \qquad Z_1 = C_1 = \mathbb{Z}[\alpha], \qquad Z_2 = Z_3 = \cdots = \{0\}$$

because both the vertex x and the edge α have zero boundary by above. In particular,

$$H_0 = \frac{\mathbb{Z}[x]}{\{0\}} = \mathbb{Z}[x], \qquad H_1 = \frac{\mathbb{Z}[\alpha]}{\{0\}} = \mathbb{Z}[\alpha]$$

and all higher homology groups are trivial.

140. One way to compute the Euler characteristic of the torus T is to note that

$$H_0(T) = \mathbb{Z}, \qquad H_1(T) = \mathbb{Z}^2, \qquad H_2(T) = \mathbb{Z}, \qquad H_3(T) = H_4(T) = \cdots = \{0\}$$

by Problem 135, whence $\chi(T) = 1 - 2 + 1 = 0$. Alternatively, one can note that

$$C_0(T) = \mathbb{Z}, \qquad C_1(T) = \mathbb{Z}^3, \qquad C_2(T) = \mathbb{Z}^2, \qquad C_3(T) = C_4(T) = \cdots = \{0\}$$

and then use the Euler-Poincaré theorem to conclude that $\chi(T) = 1 - 3 + 2 = 0$.

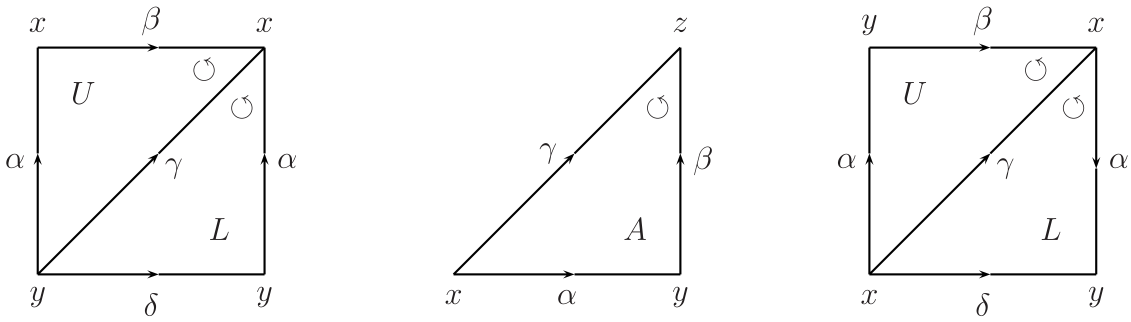

Figure 1: The cylinder; the unit disc B^2; and the Möbius strip.

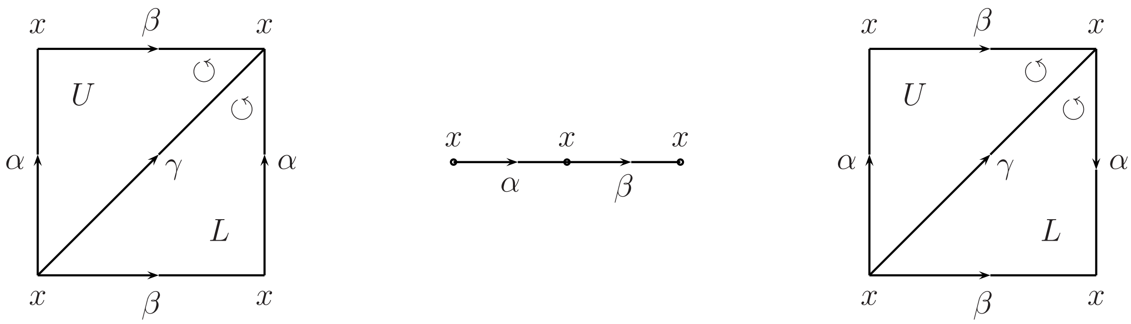

Figure 2: The torus; the figure eight; and the Klein bottle.

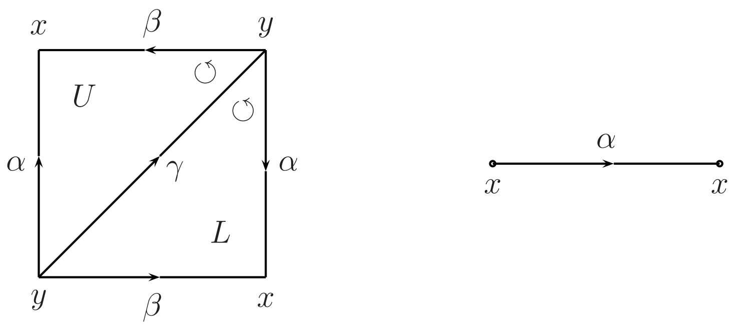

Figure 3: The projective space $\mathbb{R}P^2$ and the unit circle S^1.