Maths 212: Homework Solutions

55. Despite the original statement of this problem, one has to also assume that the given space is Hausdorff. Suppose then that X is a Hausdorff, limit point compact topological space and let $f\colon X \to Y$ be continuous.

- Assume A is an infinite subset of the image that has no limit points. In what follows, we denote the elements of A by $f(x_\alpha)$, using distinct indices for distinct elements. Since no element of A is a limit point by assumption, each $f(x_\alpha)$ has a neighbourhood U_α which contains no other $f(x_\beta)$. In particular, the inverse images $V_\alpha = f^{-1}(U_\alpha)$ are such that

$$x_\beta \in V_\alpha \iff f(x_\beta) \in U_\alpha \iff f(x_\beta) = f(x_\alpha) \iff \beta = \alpha.$$

This means that each V_α contains x_α but no other x_β. Now, consider the set

$$B = \{x_\alpha \in X \ : \ f(x_\alpha) \in A\} \subset f^{-1}(A).$$

Being an infinite subset of X, this set has a limit point x, and we must actually have

$$x \in \operatorname{Cl} B \subset \operatorname{Cl} f^{-1}(A).$$

Since A has no limit points, however, A is closed, and so is its inverse image. This gives

$$x \in f^{-1}(A) \quad \Longrightarrow \quad f(x) \in A \quad \Longrightarrow \quad f(x) = f(x_\alpha) \in U_\alpha \quad \Longrightarrow \quad x \in V_\alpha$$

for some α. Moreover, x_α is not a limit point of B, as its neighbourhood V_α contains no other x_β. In particular, $x_\alpha \neq x$ and we may now use the fact that X is Hausdorff to find a neighbourhood W of x which fails to contain x_α. This makes $V_\alpha \cap W$ a neighbourhood of x which contains no x_β at all, contrary to the fact that x is a limit point of B.

- Finally, we show that the continuous image of a limit point compact topological space X need not be limit point compact in general. To see this, we note that the set

$$\mathscr{B} = \{\ldots, \{-1, 0\}, \{1, 2\}, \{3, 4\}, \ldots\}$$

forms a basis for some topology on $X = \mathbb{Z}$. With respect to this topology, X is limit point compact because any subset containing -1 has 0 as a limit point, any subset containing 1 has 2 as a limit point, and so on. Let us now define a function $f\colon \mathbb{Z} \to 2\mathbb{Z}$ by setting

$$f(-1) = f(0) = 0, \qquad f(1) = f(2) = 2, \qquad f(3) = f(4) = 4,$$

and so on. If we equip $2\mathbb{Z}$ with the discrete topology, then f is continuous because

$$f^{-1}(2n) = \{2n - 1, 2n\}$$

is open in X for each n. However, the image $2\mathbb{Z}$ is not limit point compact because every element of $2\mathbb{Z}$ has a neighbourhood which only contains that element.

56. Suppose that A is an infinite subset of $Y \subset X$. Since X is limit point compact, A has a limit point $x \in X$. This limit point of A lies in the closure of A, so

$$x \in \operatorname{Cl} A \subset \operatorname{Cl} Y = Y$$

because $A \subset Y$ and since Y is closed. In particular, the limit point x is actually in Y.

57. Define $f\colon X \to \mathbb{R}$ by the formula $f(x, y) = x^2 + y^2 - 1$. Being a polynomial function, f is then continuous, so the composition $f \circ \gamma\colon [0, 1] \to \mathbb{R}$ is continuous as well. Moreover,

$$f(\gamma(0)) = f(x_0, y_0) = x_0^2 + y_0^2 - 1$$

is negative because (x_0, y_0) lies in the interior of the unit circle, while

$$f(\gamma(1)) = f(x_1, y_1) = x_1^2 + y_1^2 - 1$$

is positive because (x_1, y_1) lies in the exterior of the unit circle. Invoking the intermediate value property for continuous functions, we find that some $0 < t < 1$ exists such that

$$f(\gamma(t)) = 0.$$

This actually shows that $\gamma(t)$ lies on the unit circle, contrary to the fact that $\gamma(t) \in X$.

58. To show that $\delta = 1$ is a Lebesgue number for the given open cover, let U be a nonempty subset of X with diameter less than 1 and let $x \in U$ be arbitrary. Since

$$y \in U \quad \Longrightarrow \quad d(x, y) \leq \operatorname{diam} U < 1 \quad \Longrightarrow \quad y \in B_1(x),$$

we conclude that U is contained in $B_1(x)$, which is a single element of $\mathcal{U}$.

59. Suppose that X is sequentially compact and that $U_1, U_2, \dots$ form an open cover of X. If this cover has no finite subcover, then $V_n = U_1 \cup \cdots \cup U_n$ fails to cover X for each n, so we can always find a point $x_n \notin V_n$. This gives us a sequence x_n of points in X; let x_{n_k} be a convergent subsequence, say $x_{n_k} \to x$. Since the U_i's cover X, we have $x \in U_N$ for some N, and since $x_{n_k} \to x$, we also have $x_{n_k} \in U_N$ for all large enough k. Assuming k is so large that $n_k > N$, however, this leads to the contradiction $x_{n_k} \in U_N \subset V_N \subset V_{n_k}$.

60. Suppose that X is a countably compact topological space.

Step 1. We show that every closed subset $A \subset X$ is countably compact. Indeed, given a countable open cover of A, we may append $X - A$ to get a countable open cover of X. Since the latter cover has a finite subcover, however, the former one does as well.

Step 2. We show that every countable subset $A \subset X$ which has no limit points is finite. Note that such a subset is automatically closed, hence also countably compact by Step 1. Now, given a point $x \in A$, we know that x is not a limit point of A, so we can always find a neighbourhood U_x of x which contains no other point of A. Since the sets U_x form a countable open cover of A, finitely many of them will then cover A, say n. Since each of them contains exactly one element of A, we conclude that A has exactly n elements.

Step 3. We show that every infinite subset $B \subset X$ has a limit point. Pick a point $x_1 \in B$, a second point $x_2 \neq x_1$, and so on. Since B is infinite, we can proceed in this manner to obtain an infinite subset $A \subset B$ which is also countable. In view of Step 2, such a subset does have a limit point x. Being a limit point of A, however, x is also a limit point of the bigger set B. Namely, every neighbourhood of x intersects A at a point other than x, so it actually intersects B at a point other than x.

61. A countably compact metric space is limit point compact by the previous problem, so it must actually be compact as well. In other words, there is no such metric space.

62. To check reflexivity, we need to check that every element $x \in X$ lies in some connected subset of X; this is clear because x lies in the connected set $\{x\}$. To check symmetry, we need to check that x, y lie in some connected subset of X whenever y, x do; this is also clear. To check transitivity, suppose that x, y lie in the connected set A and that y, z lie in the connected set B. Since A and B have a point in common, their union $A \cup B$ is then a connected set that contains each of x, z.

- Finally, we show that every equivalence class C is connected. Suppose that $A|B$ forms a partition of C. Since the sets A, B are nonempty, we can choose points $a \in A$ and $b \in B$. Being in the same equivalence class, these points must lie in some connected set C_0. Since every element of C_0 is in the equivalence class of a, this actually implies that $C_0 \subset C$. In particular, C_0 is a connected subset of the partition, so it must lie entirely within a single part. Assuming that $C_0 \subset A$ without loss of generality, one finds that $b \in C_0 \subset A$, which is contrary to the fact that A and B are disjoint.

63. Note that f_n converges pointwise to the zero function because $f_n(1) = 0$ for all n and

$$f_n(x) = x^n(1-x) \longrightarrow 0 \qquad \text{whenever } 0 \leq x < 1.$$

To see whether the convergence is uniform, we note that

$$f_n(x) = x^n - x^{n+1} \quad \implies \quad f_n'(x) = x^{n-1}\Big(n - (n+1)x\Big).$$

This makes $f_n(x)$ increasing on $[0, \frac{n}{n+1})$ and decreasing on $(\frac{n}{n+1}, 1]$, hence

$$\sup_{0 \leq x \leq 1} |f_n(x)| = \sup_{0 \leq x \leq 1} f_n(x) = \left(\frac{n}{n+1}\right)^n \cdot \left(1 - \frac{n}{n+1}\right) \longrightarrow \frac{1}{e} \cdot 0 = 0.$$

64. Let $\varepsilon > 0$ be arbitrary. Given any points x, y in the interval $(1, 2)$, we then have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leq |x - y|$$

because $xy > 1$. This implies that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$, as needed.

65. Let $\varepsilon = 1/2$. Given any $\delta > 0$, we can choose n large enough so that

$$\left| \frac{1}{n+1} - \frac{1}{n} \right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta.$$

Since we also have

$$\left| f\left(\frac{1}{n+1} \right) - f\left(\frac{1}{n} \right) \right| = (n+1) - n = 1 > \varepsilon,$$

we may conclude that $f(x)$ is not uniformly continuous on $(0,1)$.

66. Being the uniform limit of continuous functions, f is continuous itself. Since $x_n \to x$, we must thus have $f(x_n) \to f(x)$ as well. Given $\varepsilon > 0$, this actually implies that

$$|f(x_n) - f(x)| < \varepsilon$$

for all large enough n. Moreover, the convergence $f_n \to f$ is uniform, so we also have

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in X \text{ and all large enough } n.$$

Combining the last two equations with the triangle inequality, we now find that

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < 2\varepsilon$$

for all large enough n. In particular, we find that $f_n(x_n) \to f(x)$, as needed.

67. Since the convergence $f_n \to f$ is uniform, there exists an integer N such that

$$|f_n(x) - f(x)| < 1 \qquad \text{for all } x \in X \text{ and each } n \ge N.$$

Using the triangle inequality, we then find that

$$|f(x)| \le |f(x) - f_N(x)| + |f_N(x)| < 1 + |f_N(x)| \qquad \text{for all } x \in X.$$

68. In view of the previous problem, f and g are both bounded. Let $M > 0$ be such that

$$|f(x)| \le M, \quad |g(x)| \le M \qquad \text{for all } x \in X$$

and fix some $\varepsilon > 0$. Since $f_n \to f$ uniformly, we can then find an integer N_1 such that

$$|f_n(x) - f(x)| < M \qquad \text{for all } x \in X \text{ and each } n \ge N_1.$$

For the exact same reason, we can also find an integer N_2 such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in X \text{ and each } n \ge N_2.$$

Moreover, the convergence $g_n \to g$ is also uniform, so some integer N_3 exists such that

$$|g_n(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X \text{ and each } n \ge N_3.$$

Let us now set $N = \max(N_1, N_2, N_3)$. Using the triangle inequality, we then get

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\ &\leq |f_n(x)| \cdot \varepsilon + M\varepsilon \end{aligned}$$

for all $x \in X$ and each $n \geq N$ by above. Since we also have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 2M$$

for all $x \in X$ and each $n \geq N$, this actually implies that

$$|f_n(x)g_n(x) - f(x)g(x)| < 2M\varepsilon + M\varepsilon = 3M\varepsilon$$

for all $x \in X$ and each $n \geq N$. In particular, the convergence $f_n g_n \to fg$ is uniform.

69. Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists some $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Since the sequence x_n is Cauchy, there also exists an integer N such that

$$d(x_m, x_n) < \delta \qquad \text{for all } m, n \geq N.$$

Once we now combine the last two equations, we find that

$$|f(x_m) - f(x_n)| < \varepsilon \qquad \text{for all } m, n \geq N.$$

In particular, the sequence $f(x_n)$ is also Cauchy, as needed.

70. It is clear that f_n converges pointwise to the zero function. To show that the convergence is actually uniform, it remains to show that

$$\sup_x |f_n(x)| = \sup_x \frac{|x|}{1 + nx^2}$$

tends to zero as $n \to \infty$. Since this is an even function of x, we might as well set

$$g_n(x) = \frac{x}{1 + nx^2}$$

and focus solely on points $x \geq 0$. According to the quotient rule, we then have

$$g_n'(x) = \frac{1 + nx^2 - 2nx \cdot x}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}\,.$$

This makes g_n increasing on $[0, 1/\sqrt{n}\,)$ and decreasing on $(1/\sqrt{n}, \infty)$, hence

$$\sup_x |f_n(x)| = \sup_{x \geq 0} g_n(x) = g_n(1/\sqrt{n}\,) = \frac{1}{2\sqrt{n}} \longrightarrow 0.$$

71. Let $\varepsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly, some integer N exists such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in X \text{ and each } n \geq N.$$

Since each f_n is uniformly continuous on X, there also exists some $\delta_n > 0$ such that

$$d(x, y) < \delta_n \quad \Longrightarrow \quad |f_n(x) - f_n(y)| < \varepsilon \qquad \text{for all } x, y \in X.$$

If we now assume that $d(x, y) < \delta_N$, then the triangle inequality ensures that

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\varepsilon.$$

In particular, f is also uniformly continuous on X, as needed.

72. Let X be a discrete metric space and suppose that $\{x_n\}$ is a Cauchy sequence in X. Then there exists an integer N such that

$$d(x_m, x_n) < 1 \qquad \text{for all } m, n \geq N.$$

Since d is the discrete metric, this actually means that

$$x_m = x_n \qquad \text{for all } m, n \geq N.$$

In particular, the given sequence converges to x_N because

$$d(x_n, x_N) = 0 < \varepsilon \qquad \text{for all } n \geq N \text{ and each } \varepsilon > 0.$$

73. Let A, B be the two subsets and let x_n be a Cauchy sequence in their union. Since each term of the sequence lies in either A or B, one of these sets must contain infinitely many terms. In particular, some subsequence x_{n_k} lies entirely within either A or B. Since this subsequence is also Cauchy, the completeness of A, B ensures that x_{n_k} actually converges. Being a Cauchy sequence with a convergent subsequence, the original sequence must then converge as well.

74. Since every continuous function on a compact set is bounded, A is certainly contained in the space $X = \mathscr{B}([0,1], \mathbb{R})$. Moreover, the latter space is complete, so we need only show that A is closed in it. Suppose that $f \in \operatorname{Cl} A$ and let $f_n \in A$ be a sequence of points in A such that $f_n \to f$ in the d_∞-metric. Then $f_n \to f$ uniformly, so the limit f is continuous. This shows that $f \in A$ itself, and it also establishes the desired inclusion $\operatorname{Cl} A \subset A$.

75. Since every polynomial is continuous, A is certainly contained in the space $\mathcal{C}([0,1])$ of the previous problem. Moreover, the latter space is complete, so we need only show that A is not closed in it. Said differently, it suffices to find a sequence of polynomials $f_n \in A$ that converge uniformly to a function f which is not a polynomial. Now, the polynomials

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

are known to converge uniformly to the exponential function $f(x) = e^x$. In addition, this function is not a polynomial because $f' = f$, whereas the derivative of any polynomial is some other polynomial of lower degree.

76. Given any points x, y in the unit interval $(0,1)$, one easily finds that

$$|x^2 - y^2| = |x + y| \cdot |x - y| \leq 2 \cdot |x - y|.$$

In particular, one finds that $f(x) = x^2/3$ is a contraction on $(0,1)$ because

$$|f(x) - f(y)| = \frac{|x^2 - y^2|}{3} \leq \frac{2}{3} \cdot |x - y|$$

for all $x, y \in (0,1)$ by above. To see that f fails to have a fixed point, we note that

$$f(x) = x \quad \Longrightarrow \quad \frac{x^2}{3} = x \quad \Longrightarrow \quad x = 0, 3 \quad \Longrightarrow \quad x \notin (0,1).$$

Finally, this example does not violate Banach's theorem because $(0,1)$ is not complete.

77. Since X is complete, it suffices to show that T is a contraction. Once we note that

$$|T(f)(x) - T(g)(x)| \leq \int_0^x |f(t) - g(t)|\, dt \leq x \cdot d_\infty(f, g)$$

for each $0 \leq x \leq 1/2$, we may immediately deduce the desired

$$d_\infty(T(f), T(g)) \leq \frac{1}{2} \cdot d_\infty(f, g).$$

78. Being continuous on a compact set, f' is also bounded, say

$$|f'(s)| \leq K \qquad \text{for all } 0 \leq s \leq 1.$$

Using the fundamental theorem of calculus, we then find that

$$|f(x) - f(y)| = \left| \int_y^x f'(s)\, ds \right| \leq K|x - y|.$$

79. Indeed, $\mathbb{R}$ is complete and the function $f(x) = \frac{x^2}{1+x^2}$ is continuous, yet the image

$$f(\mathbb{R}) = [0, 1)$$

is not complete because it is not closed in $\mathbb{R}$.

80. The sequence of open intervals $A_n = (0, 1/n)$ is easily seen to be such.

81. The sequence of closed intervals $B_n = [n, \infty)$ is easily seen to be such.

82. Given an infinite discrete metric space X, one has

$$d(x, y) \leq 1 \qquad \text{for all } x, y \in X$$

and this shows that X is bounded. On the other hand, X has no finite 1-net because

$$B_1(x) = \{y \in X \,:\, d(x, y) < 1\} = \{x\},$$

so that finitely many balls of radius 1 may only cover finitely many elements of X.

83. Suppose that $f(x_n)$ is a sequence in the image. Since X is totally bounded by assumption, the sequence x_n does have a Cauchy subsequence x_{n_k}. According to Problem 69 then, the corresponding subsequence $f(x_{n_k})$ must also be Cauchy. This shows that every sequence in $f(X)$ has a Cauchy subsequence, which implies that $f(X)$ is totally bounded.

84. Let $\varepsilon = 1$. Given any $\delta > 0$, we can choose n large enough so that $x_n = 1/n$ satisfies

$$|x_n - 0| = 1/n < \delta.$$

Since we also have

$$|f_n(x_n) - f_n(0)| = (n + 1/n)^2 - n^2 = 2 + \frac{1}{n^2} > 2 > \varepsilon,$$

the sequence $f_n(x)$ is not equicontinuous at the origin.

85. Let $\varepsilon = 1$. Given any $\delta > 0$, we can choose n large enough so that $x_n = \pi/n$ satisfies

$$|x_n - 0| = \pi/n < \delta.$$

Since we also have

$$|f_n(x_n) - f_n(0)| = |\cos\pi - \cos 0| = 2 > \varepsilon,$$

the sequence $f_n(x)$ is not equicontinuous at the origin.

86. Since $A_1 = X$ is compact, its continuous image $A_2 = f(A_1)$ is compact with

$$A_2 = f(A_1) = f(X) \subset X = A_1.$$

Using the exact same argument, one finds that $A_3 = f(A_2)$ is compact with

$$A_2 \subset A_1 \quad \Longrightarrow \quad f(A_2) \subset f(A_1) \quad \Longrightarrow \quad A_3 \subset A_2.$$

Thus, we may proceed in this manner to obtain a nested sequence of nonempty, compact subsets of X. Being compact in the Hausdorff space X, these sets are all closed in X, so their intersection $A = A_1 \cap A_2 \cap \cdots$ is closed as well. Moreover, A is nonempty due to the intersection property of compact spaces. Once we now note that the image

$$f(A) = \bigcap_{n=1}^{\infty} f(A_n) = \bigcap_{n=1}^{\infty} A_{n+1} = \bigcap_{n=2}^{\infty} A_n$$

lies in X, we may conclude that

$$f(A) = f(A) \cap X = f(A) \cap A_1 = \bigcap_{n=1}^{\infty} A_n = A.$$

87a. Suppose that $x \in \operatorname{Cl} A$ and let $x_n \in A$ be such that $x_n \to x$. Then the inequality

$$\rho(x) = \inf_{z \in A} d(x, z) \leq d(x, x_n)$$

must hold for each $n \geq 1$, so we may let $n \to \infty$ to deduce that

$$\rho(x) \leq 0.$$

Since we also have $\rho(x) \geq 0$ by definition, this actually implies that $\rho(x) = 0$.

- Suppose now that $\rho(x) = 0$. Given any positive integer n, we then have

$$\inf_{z \in A} d(x, z) = \rho(x) = 0 < 1/n,$$

so $1/n$ is not a lower bound for $d(x, z)$. This means that some $z_n \in A$ is such that

$$d(x, z_n) < 1/n.$$

Once we now note that $z_n \to x$ by the last equation, we may conclude that $x \in \operatorname{Cl} A$.

87b. Let $x, y \in X$ be arbitrary. Given any $z \in A$, we must then have

$$\rho(x) \leq d(x, z) \leq d(x, y) + d(y, z)$$

by the triangle inequality. Taking the infimum over all $z \in A$, we thus find

$$\rho(x) \leq d(x, y) + \rho(y) \quad \implies \quad \rho(x) - \rho(y) \leq d(x, y).$$

Since the same argument applies to give $\rho(y) - \rho(x) \leq d(y, x)$, we actually have

$$|\rho(x) - \rho(y)| \leq d(x, y).$$

88. Define the function $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by the formula

$$T(f)(x) = \int_0^x (x - y)\, f(y)\, dy \qquad \text{for each } 0 \leq x \leq 1.$$

Then we have

$$\begin{aligned} |T(f)(x) - T(g)(x)| &\leq \int_0^x (x - y) \cdot |f(y) - g(y)|\, dy \\ &\leq d_\infty(f, g) \int_0^x (x - y)\, dy \end{aligned}$$

and we may use the substitution $u = x - y$ to get

$$\int_0^x (x - y)\, dy = \int_0^x u\, du = \frac{x^2}{2} \leq \frac{1}{2}\,.$$

Once we now combine the last two equations, we arrive at

$$|T(f)(x) - T(g)(x)| \leq \frac{1}{2} \cdot d_\infty(f, g) \quad \implies \quad d_\infty(T(f), T(g)) \leq \frac{1}{2} \cdot d_\infty(f, g)\,.$$

This shows that T is a contraction, so T has a unique fixed point by Banach's theorem. Since the zero function is clearly a fixed point, there is no other fixed point, indeed.

89. Suppose that f is not constant. Then there exists some $a \in \mathbb{R}$ such that

$$\varepsilon = \frac{|f(a) - f(0)|}{2}$$

is positive. Given $\delta > 0$, let us now choose n large enough so that $x_n = a/n$ satisfies

$$|x_n - 0| = |a|/n < \delta.$$

Since we also have

$$|f_n(x_n) - f_n(0)| = |f(a) - f(0)| = 2\varepsilon > \varepsilon,$$

the sequence f_n cannot be equicontinuous at the origin, as needed.

90. Note that the given sequence is bounded by a single constant, since

$$|F_n(x)| \leq \int_0^x |f_n(s)|\, ds \leq \int_0^x ds = x \leq 1$$

for all $0 \leq x \leq 1$ and each $n \geq 1$. In addition, the inequality

$$|F_n(x) - F_n(y)| = \left| \int_y^x f_n(s)\, ds \right| \leq |x - y|$$

implies that the sequence F_n is equicontinuous, as it certainly implies that

$$|x - y| < \varepsilon \quad \Longrightarrow \quad |F_n(x) - F_n(y)| \leq |x - y| < \varepsilon.$$

In view of the corollary to the Arzela-Ascoli theorem, this also completes the proof.

91. Since the sequence f_n is equicontinuous on a compact set, it is uniformly equicontinuous. Given $\varepsilon > 0$, we may thus find some $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad |f_n(x) - f_n(y)| < \varepsilon \qquad \text{for all } x, y \in X \text{ and each } n \geq 1.$$

Since X is compact, X is totally bounded as well, so it has a finite δ-net, say $y_1, \ldots, y_j$. By assumption, the sequence $f_n(y_i)$ converges to $f(y_i)$ for each i, so this sequence is also Cauchy for each i. In particular, we can always find an integer N_i such that

$$|f_n(y_i) - f_m(y_i)| < \varepsilon \qquad \text{for all } m, n \geq N_i.$$

Letting N denote the maximum of the finitely many N_i's, we thus arrive at

$$|f_n(y_i) - f_m(y_i)| < \varepsilon \qquad \text{for all } m, n \geq N \text{ and each } 1 \leq i \leq j.$$

Suppose now that $x \in X$ is arbitrary. Since $d(x, y_i) < \delta$ for some i, we also have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y_i)| + |f_n(y_i) - f_m(y_i)| + |f_m(y_i) - f_m(x)| < 3\varepsilon$$

for all $m, n \geq N$ by above. This shows that the sequence f_n is uniformly Cauchy, which also implies it is uniformly convergent.

92. Fix some $\varepsilon > 0$. Since $f_n \to f$ uniformly, some integer N exists such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } \ 0 \leq x \leq 1 \text{ and each } n \geq N.$$

As long as $n \geq N$, in particular, we must also have

$$\left| \int_0^1 f_n(x)\,dx - \int_0^1 f(x)\,dx \right| = \left| \int_0^1 [f_n(x) - f(x)]\,dx \right| < \int_0^1 \varepsilon\,dx = \varepsilon.$$

93. Let $\varepsilon = 1/2$. Given any $\delta > 0$, we can choose n large enough so that $x_n = 1/n$ satisfies

$$|x_n - 0| = 1/n < \delta.$$

Since we also have

$$|f_n(x_n) - f_n(0)| = |1 - 0| > \varepsilon,$$

the functions f_n do not form an equicontinuous family at the origin.

94. To see that F is closed, suppose $f \in \operatorname{Cl} F$ and let $f_n \in F$ be such that $f_n \to f$ uniformly. Then f is the uniform limit of continuous functions, hence also continuous. Moreover,

$$|f_n(x) - f_n(y)| \leq |x - y| \qquad \text{for all } 0 \leq x, y \leq 1$$

so we can let $n \to \infty$ to find that

$$|f(x) - f(y)| \leq |x - y| \qquad \text{for all } 0 \leq x, y \leq 1.$$

And since $f_n(0) = 0$ for all n, we must similarly have $f(0) = 0$ as well.

- To see that F is bounded, we note that each $f \in F$ is such that

$$|f(x) - f(0)| \leq |x - 0| \quad \Longrightarrow \quad |f(x)| \leq |x| \leq 1 \qquad \text{for all } 0 \leq x \leq 1.$$

- To see that F is equicontinuous, we let $\varepsilon > 0$ be arbitrary and note that

$$|x - y| < \varepsilon \quad \Longrightarrow \quad |f(x) - f(y)| \leq |x - y| < \varepsilon \qquad \text{for all } f \in F.$$

Since F is closed, bounded and equicontinuous, it is also compact by Arzela-Ascoli.