Maths 212: Homework Solutions

1. The definition of A ensures that $x \leq \pi$ for all $x \in A$, so π is an upper bound of A. To show it is the least upper bound, suppose $x_* < \pi$ and consider two cases.

   - If $x_* < 1$, then x_* cannot be an upper bound of A because $1 \in A$.
   - If $1 \leq x_* < \pi$, then we can choose a rational y with $x_* < y < \pi$, and x_* cannot be an upper bound of A because $y \in A$.

   Since no real number $x_* < \pi$ can be an upper bound of A, we deduce that $\sup A = \pi$.

2. Since A is bounded by assumption, both $\inf A$ and $\sup A$ exist. Moreover, we have

   $$\inf A \leq x \leq \sup A \qquad \text{for all } x \in A.$$

   Since the elements of B are also elements of A, this actually implies

   $$\inf A \leq x \leq \sup A \qquad \text{for all } x \in B.$$

   Now, the last equation makes $\sup A$ an upper bound of B, while $\sup B$ is the least upper bound of B. In particular, it must be the case that $\sup A \geq \sup B$. Similarly, $\inf A$ is a lower bound of B by above, so it must be the case that $\inf A \leq \inf B$.

3. **Step 1.** Using induction on n, one can easily show that $s_n > 0$ for all n.

   Step 2. We claim that $s_n < \sqrt{2}$ for all n. This is the case when $n = 1$ because $s_1 = 1$. Suppose it is the case for some n. Since s_n is positive by Step 1, we then have

   $$\begin{aligned} s_{n+1} < \sqrt{2} &\iff 2s_n + 2 < (s_n + 2)\sqrt{2} \\ &\iff (2 - \sqrt{2}\,)s_n < 2\sqrt{2} - 2 = (2 - \sqrt{2}\,)\sqrt{2} \\ &\iff s_n < \sqrt{2}. \end{aligned}$$

   As the last inequality holds by the induction hypothesis, the first one does as well.

   Step 3. We show that $\{s_n\}$ is increasing. Since s_n is positive by Step 1, we have

   $$\begin{aligned} s_{n+1} > s_n \quad &\iff \quad 2s_n + 2 > (s_n + 2)s_n = s_n^2 + 2s_n \\ &\iff \quad 2 > s_n^2 \\ &\iff \quad \sqrt{2} > s_n. \end{aligned}$$

   As the last inequality holds by Step 2, the first one does as well.

   Step 4. We already know by above that $\{s_n\}$ is increasing and bounded. This implies that the given sequence is convergent. Once we now denote the limit by L, we find

   $$s_{n+1} = \frac{2s_n + 2}{s_n + 2} \quad \Longrightarrow \quad L = \frac{2L + 2}{L + 2} \quad \Longrightarrow \quad L^2 + 2L = 2L + 2.$$

   Since the limit of a non-negative sequence is also non-negative, this actually gives

   $$L^2 + 2L = 2L + 2 \quad \Longrightarrow \quad L^2 = 2 \quad \Longrightarrow \quad L = \sqrt{2}.$$

4. Set $\varepsilon = L - L'$. Then $\varepsilon > 0$, so there exists an integer N such that

$$|s_n - L| < \varepsilon = L - L' \qquad \text{for all } n \geq N.$$

Since this implies

$$L - s_n \leq |L - s_n| < L - L' \qquad \text{for all } n \geq N,$$

we deduce that

$$s_n > L' \qquad \text{for all } n \geq N.$$

5. Let $\varepsilon > 0$ be arbitrary. Since $a_n \to L$, there exists an integer N_1 such that

$$|a_n - L| < \varepsilon \qquad \text{for all } n \geq N_1.$$

Rewrite the last equation in the form

$$L - \varepsilon < a_n < L + \varepsilon \qquad \text{for all } n \geq N_1.$$

Since $c_n \to L$, a similar argument gives us an integer N_2 such that

$$L - \varepsilon < c_n < L + \varepsilon \qquad \text{for all } n \geq N_2.$$

Letting $N = \max(N_1, N_2)$, we now find

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \quad \Longrightarrow \quad |b_n - L| < \varepsilon$$

for all $n \geq N$. Since $\varepsilon > 0$ was arbitrary, this actually shows that $b_n \to L$.

6. Let $\varepsilon > 0$ be arbitrary. Since the given sequence is convergent, it is also Cauchy. Thus, there exists an integer N such that

$$|s_m - s_n| < \varepsilon \qquad \text{for all } m, n \geq N.$$

Taking $m = n + 1$, as we may, we now find

$$|s_{n+1} - s_n| < \varepsilon \qquad \text{for all } n \geq N.$$

Since $\varepsilon > 0$ was arbitrary, this actually shows that $s_{n+1} - s_n \to 0$.

7. Let x be a real number and let $\{x_n\}$ be a sequence with $x_n \to x$. When it comes to the rational terms of the sequence, we have $f(x_n) = x_n \to x$. When it comes to the irrational ones, we have $f(x_n) = 0$. In order for f to be continuous at x, these two quantities must agree with one another and they must also agree with $f(x)$. In particular, the point $x = 0$ is the only point at which f is continuous.

8. First, we check property (M1). Since $d(x, y) \geq 0$ and since $1 > 0$, we have

$$d_0(x, y) = \min\{1, d(x, y)\} \geq 0$$

as well. Moreover, the fact that $1 \neq 0$ implies

$$d_0(x, y) = 0 \iff d(x, y) = 0 \iff x = y$$

because d is a metric. Next, we prove property (M2). Since d is a metric, we have

$$d_0(y, x) = \min\{1, d(y, x)\} = \min\{1, d(x, y)\} = d_0(x, y).$$

Finally, we check property (M3). Given points $x, y, z \in X$, we have to check that

$$\min\{1, d(x, z)\} \leq \min\{1, d(x, y)\} + \min\{1, d(y, z)\}.$$

If either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, this inequality holds because the left hand side is at most 1 and the right hand side is at least 1. Otherwise, $d(x, y) < 1$ and $d(y, z) < 1$, so the triangle inequality for d gives

$$\begin{aligned} \min\{1, d(x, z)\} &\leq d(x, z) \leq d(x, y) + d(y, z) \\ &= \min\{1, d(x, y)\} + \min\{1, d(y, z)\}. \end{aligned}$$

9. First, we check property (M1). Since $d((x_1, y_1), (x_2, y_2))$ is defined as the square root of a certain expression, it is certainly non-negative. Moreover, we have

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) = 0 &\iff d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2 = 0 \\ &\iff x_1 = x_2 \quad \text{and} \quad y_1 = y_2 \\ &\iff (x_1, y_1) = (x_2, y_2). \end{aligned}$$

Next, we check property (M2). Since d_1 and d_2 are known to be metrics, we have

$$\begin{aligned} d((x_2, y_2), (x_1, y_1)) &= \sqrt{d_1(x_2, x_1)^2 + d_2(y_2, y_1)^2} \\ &= \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2} \\ &= d((x_1, y_1), (x_2, y_2)). \end{aligned}$$

Finally, we check property (M3). For ease of notation, it is convenient to set

$$A_1 = d_1(x_1, x_2), \qquad B_1 = d_1(x_2, x_3), \qquad C_1 = d_1(x_1, x_3),$$

$$A_2 = d_2(y_1, y_2), \qquad B_2 = d_2(y_2, y_3), \qquad C_2 = d_2(y_1, y_3).$$

Then the triangle inequality for d_i implies that

$$C_i \leq A_i + B_i \quad \Longrightarrow \quad C_i^2 \leq A_i^2 + B_i^2 + 2A_iB_i$$

for $i = 1, 2$. Adding these inequalities and using Cauchy-Schwarz, we then get

$$\begin{aligned} C_1^2 + C_2^2 &\leq A_1^2 + A_2^2 + B_1^2 + B_2^2 + 2\sum_{i=1}^{2} A_i B_i \\ &\leq A_1^2 + A_2^2 + B_1^2 + B_2^2 + 2\sqrt{A_1^2 + A_2^2}\,\sqrt{B_1^2 + B_2^2}. \end{aligned}$$

Once we now take square roots of both sides, we arrive at

$$\sqrt{C_1^2 + C_2^2} \leq \sqrt{A_1^2 + A_2^2} + \sqrt{B_1^2 + B_2^2}.$$

Since this is precisely the desired triangle inequality for d, the proof is complete.

10. Suppose there is an element $z \in B_{r/2}(x) \cap B_{r/2}(y)$. Then that element satisfies

$$d(x, z) < \frac{r}{2}, \qquad d(y, z) < \frac{r}{2}.$$

According to the triangle inequality, we must thus have

$$r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r.$$

Since this is a contradiction, however, we deduce that $B_{r/2}(x) \cap B_{r/2}(y) = \varnothing$.

11. Pick an element $y \in U$. Then $\varepsilon = d(x, y) - r$ is positive. We claim that $B_\varepsilon(y)$ is an open ball around y that lies entirely within U. Indeed, we have

$$\begin{aligned} z \in B_\varepsilon(y) \quad &\Longrightarrow \quad d(z, y) < \varepsilon \\ &\Longrightarrow \quad d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \varepsilon \\ &\Longrightarrow \quad d(x, z) > d(x, y) - \varepsilon = r \\ &\Longrightarrow \quad z \in U. \end{aligned}$$

12. The open ball $B_1((0, 0))$ consists of all points (x_1, x_2) with

$$d_\infty((x_1, x_2), (0, 0)) = \max\{|x_1|, |x_2|\} < 1.$$

These are precisely the points with $|x_1| < 1$ and $|x_2| < 1$. In particular, the desired open ball is the interior of a square whose vertices are located at $(\pm 1, 1)$ and $(\pm 1, -1)$.

13. Let S be a subset of a discrete metric space X and pick some $s \in S$. Then $B_1(s)$ is an open ball around s that lies entirely within S because $B_1(s) = \{s\} \subset S$.

14. First of all, it is clear that

$$\max\{|x_1 - y_1|, |x_2 - y_2|\}^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2;$$

taking square roots of both sides, we then find

$$d_\infty((x_1,x_2),(y_1,y_2)) \leq d_2((x_1,x_2),(y_1,y_2)).$$

Also, it is easy to establish the inequality

$$|x_1-y_1|^2+|x_2-y_2|^2 \leq \Big(|x_1-y_1|+|x_2-y_2|\Big)^2$$

by expanding the right hand side; this gives

$$d_2((x_1,x_2),(y_1,y_2)) \leq d_1((x_1,x_2),(y_1,y_2)).$$

Finally, we have

$$\begin{aligned} |x_1-y_1| &\leq \max\{|x_1-y_1|,|x_2-y_2|\}, \\ |x_2-y_2| &\leq \max\{|x_1-y_1|,|x_2-y_2|\}. \end{aligned}$$

Adding these two inequalities, we then find

$$d_1((x_1,x_2),(y_1,y_2)) \leq 2\cdot d_\infty((x_1,x_2),(y_1,y_2)).$$

Since $d_\infty \leq d_2 \leq d_1 \leq 2\cdot d_\infty$ by above, all these metrics are Lipschitz equivalent.

15i. The set $U=[0,2)$ is not open in $\mathbb{R}$ with the usual metric.

15ii. The set $U=[0,2)$ is open in $[0,3]$ because $U=(-2,2)\cap[0,3]$.

15iii. The set $U=\{0,2\}$ is not open in $\mathbb{R}$ with the usual metric.

15iv. In a discrete metric space, all sets are open.

16. Suppose U is open in Z. Then $g^{-1}(U)$ is open in Y, so $f^{-1}(g^{-1}(U))$ is open in X. On the other hand, it is easy to see that

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= \{x\in X \,:\, f(x)\in g^{-1}(U)\} = \{x\in X \,:\, g(f(x))\in U\} \\ &= (g\circ f)^{-1}(U). \end{aligned}$$

Since this set is open in X by above, we deduce that $g\circ f$ is continuous.

17. Suppose U is open in Y and consider two cases.

- If $y_0\in U$, then $f^{-1}(U)$ is the whole space X, which is open in X.
- If $y_0\notin U$, then $f^{-1}(U)$ is the empty set, which is open in X as well.

In any case then, $f^{-1}(U)$ is open in X, so f is continuous.

18. Suppose U is a union of open balls. Then U is a union of open sets, hence also open.

Conversely, suppose that U is open. For each element $x \in U$, we can then find an open ball $B_\varepsilon(x)$ which lies entirely within U. Since this implies

$$U = \bigcup_{x\in U} \{x\} \subset \bigcup_{x\in U} B_\varepsilon(x) \subset U,$$

the sets appearing above must all be equal. In particular, U is a union of open balls.

19. Pick an element $(x_1, x_2) \in S$. Our goal is to find an open ball $B_\varepsilon((x_1, x_2))$ that lies entirely within S. We claim that such an open ball is provided for the choice

$$\varepsilon = \min_{i=1,2} \{x_i, 1 - x_i\}.$$

Note that ε is positive because

$$\begin{aligned} (x_1, x_2) \in S \quad &\Longrightarrow \quad 0 < x_i < 1 \qquad &&\text{for } i = 1, 2 \\ &\Longrightarrow \quad 0 < 1 - x_i < 1 \qquad &&\text{for } i = 1, 2. \end{aligned}$$

Now, let $(y_1, y_2) \in B_\varepsilon((x_1, x_2))$ be arbitrary. Then we have

$$\left[\sum_{i=1}^{2} |y_i - x_i|^2\right]^{1/2} < \varepsilon \quad \Longrightarrow \quad \sum_{i=1}^{2} |y_i - x_i|^2 < \varepsilon^2,$$

so we also have

$$|y_i - x_i|^2 \leq \sum_{i=1}^{2} |y_i - x_i|^2 < \varepsilon^2 \quad \Longrightarrow \quad |y_i - x_i| < \varepsilon$$

for $i = 1, 2$. In view of the definition of ε, this actually implies that

$$0 \leq x_i - \varepsilon < y_i < x_i + \varepsilon \leq 1 \quad \Longrightarrow \quad 0 < y_i < 1$$

for $i = 1, 2$. In particular, it implies that $(y_1, y_2) \in S$, as needed.

20. Suppose U is open in Y and consider its inverse image

$$\begin{aligned} f|_A^{-1}(U) &= \{x \in A \,:\, f|_A(x) \in U\} = \{x \in A \,:\, f(x) \in U\} \\ &= \{x \in A \,:\, x \in f^{-1}(U)\} = f^{-1}(U) \cap A. \end{aligned}$$

Since $f^{-1}(U)$ is open in X, we see that $f|_A^{-1}(U)$ is open in A. Thus, $f|_A$ is continuous.

21. Suppose U is open in Y and consider its inverse image

$$i^{-1}(U) = \{x \in X \,:\, i(x) \in U\} = \{x \in X \,:\, x \in U\} = U \cap X.$$

Since U is open in Y, we see that $i^{-1}(U)$ is open in X. Thus, i is continuous.

22a. To check property (B1), it suffices to note that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-\infty, n)$$

is a union of elements of $\mathscr{B}$. To check property (B2), it suffices to note that

$$(-\infty, a) \cap (-\infty, b) = (-\infty, \min(a, b)) \in \mathscr{B}.$$

22b. Let $B = (-\infty, a)$ be a basis element for the given topology and let $x \in B$. Since

$$x \in (x - 1, a) \subset B,$$

the usual topology on $\mathbb{R}$ is finer.

23a. Suppose X has the discrete topology and let U be open in Y. The inverse image of U is then a subset of X, so it is automatically open in X. This shows that f is continuous.

23b. Suppose Y has the indiscrete topology. The only open sets in Y are then the empty set and the whole space Y. The inverse image of the former is the empty set and this is open in X. The inverse image of the latter is X and this is open in X as well.

24. Take an element of the subspace topology, say $U \cap Y$, where U is open in X. Since $\mathscr{B}_X$ is a basis for the topology on X, we can write U as a union of elements of $\mathscr{B}_X$. This gives

$$U = \bigcup B_\alpha \quad \Longrightarrow \quad U \cap Y = \left(\bigcup B_\alpha\right) \cap Y = \bigcup (B_\alpha \cap Y),$$

whence $U \cap Y$ is a union of elements of $\mathscr{B}_Y$, as needed.

25. Note that the given set is merely the unit interval $(0, 1)$ with the points $1/2, 1/3, 1/4, \ldots$ removed. It is easy to see that we can express this set in the form

$$A = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right).$$

Being the union of open intervals, A must thus be open itself.

26. Suppose that U is open in Y. Then we can write U as a union of elements $B_\alpha \in \mathscr{B}_Y$. As the inverse image of each B_α is open in X by assumption, we see that

$$f^{-1}(U) = f^{-1}\left(\bigcup B_\alpha\right) = \bigcup f^{-1}(B_\alpha)$$

is open in X as well. This shows that f is continuous.

27. Let $\mathscr{T}_u$ and $\mathscr{T}_p$ denote the usual and product topology on $\mathbb{R}^2$, respectively.

- ▶ First, we show that $\mathscr{T}_u \supset \mathscr{T}_p$. Suppose that $(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2)$ and let

$$\varepsilon = \min_{i=1,2} \{x_i - a_i,\ b_i - x_i\}.$$

Then $\varepsilon > 0$ and so the open ball $B_\varepsilon((x_1, x_2))$ is a basis element for the usual topology. In addition, we have

$$\begin{aligned} (y_1, y_2) \in B_\varepsilon((x_1, x_2)) \quad &\Longrightarrow \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 < \varepsilon^2 \\ &\Longrightarrow \quad a_i \leq x_i - \varepsilon < y_i < x_i + \varepsilon \leq b_i \qquad \text{for } i = 1, 2 \\ &\Longrightarrow \quad (y_1, y_2) \in (a_1, b_1) \times (a_2, b_2). \end{aligned}$$

Since this implies that $B_\varepsilon((x_1, x_2)) \subset (a_1, b_1) \times (a_2, b_2)$, we conclude that $\mathscr{T}_u \supset \mathscr{T}_p$.

- ▶ Next, we show that $\mathscr{T}_p \supset \mathscr{T}_u$. Suppose that $(x_1, x_2) \in B_\varepsilon((a_1, a_2))$ and note that

$$\delta = \frac{1}{\sqrt{2}} \left(\varepsilon - \sqrt{\sum_{i=1}^{2} |x_i - a_i|^2} \right)$$

is positive. A basis element for the product topology on $\mathbb{R}^2$ is then

$$U = (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta)$$

and it suffices to show that $U \subset B_\varepsilon((a_1, a_2))$. Now, given some $(y_1, y_2) \in U$, we have

$$|y_i - x_i| < \delta \quad \Longrightarrow \quad |y_i - a_i| \leq |y_i - x_i| + |x_i - a_i| < \delta + |x_i - a_i|$$

for $i = 1, 2$. Squaring both sides of this inequality and summing, we then find

$$\sum_{i=1}^{2} |y_i - a_i|^2 < \sum_{i=1}^{2} \delta^2 + \sum_{i=1}^{2} |x_i - a_i|^2 + 2 \sum_{i=1}^{2} \delta\, |x_i - a_i|.$$

In view of the Cauchy-Schwarz inequality, this also gives

$$\sum_{i=1}^{2} |y_i - a_i|^2 < \sum_{i=1}^{2} \delta^2 + \sum_{i=1}^{2} |x_i - a_i|^2 + 2 \sqrt{\sum_{i=1}^{2} \delta^2}\, \sqrt{\sum_{i=1}^{2} |x_i - a_i|^2}.$$

Once we now note that the right hand side is a perfect square, we arrive at

$$\sqrt{\sum_{i=1}^{2} |y_i - a_i|^2} < \sqrt{\sum_{i=1}^{2} \delta^2} \; + \; \sqrt{\sum_{i=1}^{2} |x_i - a_i|^2} = \varepsilon.$$

Since this implies that $U \subset B_\varepsilon((a_1, a_2))$, we conclude that $\mathscr{T}_p \supset \mathscr{T}_u$.

28. Since the open intervals (a,b) form a basis for the usual topology on $\mathbb{R}$, it suffices to check that the inverse image

$$d^{-1}((a,b)) = \{(x,y) \in X \times X \,:\, a < d(x,y) < b\}$$

is open in $X \times X$ whenever $a < b$. Pick some $(x,y) \in d^{-1}((a,b))$ and set

$$\varepsilon = \min\Big\{d(x,y) - a,\; b - d(x,y)\Big\}.$$

Then $\varepsilon > 0$ and it suffices to check that $B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y) \subset d^{-1}((a,b))$. Suppose then that $x' \in B_{\varepsilon/2}(x)$ and $y' \in B_{\varepsilon/2}(y)$. According to the triangle inequality, we have

$$\begin{aligned} d(x',y') &\leq d(x',x) + d(x,y) + d(y,y') \\ &< \varepsilon/2 + d(x,y) + \varepsilon/2 = d(x,y) + \varepsilon \leq b \end{aligned}$$

as well as

$$\begin{aligned} a + \varepsilon \leq d(x,y) &\leq d(x,x') + d(x',y') + d(y',y) \\ &< \varepsilon/2 + d(x',y') + \varepsilon/2 = d(x',y') + \varepsilon. \end{aligned}$$

This shows that $a < d(x',y') < b$, hence also $(x',y') \in d^{-1}((a,b))$, as needed.

29. Define a function $f\colon X \times Y \to Y \times X$ by the formula $f(x,y) = (y,x)$. Then f is clearly bijective and equal to its inverse. To check that f is a homeomorphism, it suffices to check continuity. Suppose U is open in X and V is open in Y. Then $V \times U$ is one of the basis elements in the product topology of $Y \times X$, while

$$\begin{aligned} f^{-1}(V \times U) &= \{(x,y) \in X \times Y \,:\, f(x,y) \in V \times U\} \\ &= \{(x,y) \in X \times Y \,:\, (y,x) \in V \times U\} = U \times V. \end{aligned}$$

This shows that the inverse image of each basis element is open. Thus, f is continuous.

30a. We have $\operatorname{Int} A = \varnothing$ and $\operatorname{Cl} A = \mathbb{R} \times [0,\infty)$.

30b. We have $\operatorname{Int} B = B$ and $\operatorname{Cl} B = \mathbb{R}^2$.

31. If $x \in \operatorname{Cl} A$, then every neighbourhood of x intersects A, hence every neighbourhood of x intersects the larger set B. This proves the desired inclusion $\operatorname{Cl} A \subset \operatorname{Cl} B$.

32. Since the closures $\operatorname{Cl} A$ and $\operatorname{Cl}(X - A)$ are known to be closed, their intersection

$$\operatorname{Bd} A = \operatorname{Cl} A \cap \operatorname{Cl}(X - A)$$

must be closed as well.

33. Let $A = \{a_1, \ldots, a_n\}$ be a finite subset of a metric space X and let $x \in X$ be arbitrary.

- In the case that $x \notin A$, we can set
$$\varepsilon = \min\{d(x, a_1), \ldots, d(x, a_n)\}$$
to obtain an open ball $B_\varepsilon(x)$ around x that does not intersect A at all.
- In the case that $x \in A$, we have $x = a_i$ for some i, so we can take
$$\varepsilon = \min_{j \neq i} d(a_i, a_j)$$
to obtain an open ball $B_\varepsilon(x)$ that intersects A only at the point $x = a_i$.

In either case then, there exists some neighbourhood of x which fails to intersect A at a point other than x. This means that no point $x \in X$ can be a limit point of A.

▶ To see that A is closed, we note that $\operatorname{Cl} A$ is the union of A and its limit points. Since A has no limit points by above, this gives $\operatorname{Cl} A = A$ and so A is closed.

34. One such subset is $\mathbb{Q}$ because
$$\operatorname{Int}(\operatorname{Cl} \mathbb{Q}) = \operatorname{Int} \mathbb{R} = \mathbb{R}, \qquad \operatorname{Cl}(\operatorname{Int} \mathbb{Q}) = \operatorname{Cl} \varnothing = \varnothing.$$
Another such subset is $A = (0, 1) \cup (1, 2)$ because
$$\operatorname{Int}(\operatorname{Cl} A) = \operatorname{Int}[0, 2] = (0, 2), \qquad \operatorname{Cl}(\operatorname{Int} A) = \operatorname{Cl} A = [0, 2].$$

35. First, suppose that f is continuous and let $A \subset X$. Then $\operatorname{Cl} f(A)$ is closed in Y, so the inverse image of this set is closed in X. In addition, we have
$$A \subset f^{-1}(f(A)) \subset f^{-1}(\operatorname{Cl} f(A)),$$
so $f^{-1}(\operatorname{Cl} f(A))$ is actually a closed set containing A. Since $\operatorname{Cl} A$ is the smallest closed set with this property, we conclude that $\operatorname{Cl} A \subset f^{-1}(\operatorname{Cl} f(A))$.

▶ Next, suppose that $\operatorname{Cl} A \subset f^{-1}(\operatorname{Cl} f(A))$ for each $A \subset X$ and assume U is closed in Y. Then $A = f^{-1}(U)$ is a subset of X, so it must be the case that
$$\operatorname{Cl} f^{-1}(U) \subset f^{-1}(\operatorname{Cl} U) = f^{-1}(U) \subset \operatorname{Cl} f^{-1}(U).$$
In particular, all these sets must be equal and so $f^{-1}(U) = \operatorname{Cl} f^{-1}(U)$ is closed in X.

36. To see that $\operatorname{Int} A \cap \operatorname{Bd} A = \varnothing$, we note that
$$\begin{aligned} x \in \operatorname{Int} A \quad &\Longrightarrow \quad \text{some neighbourhood of } x \text{ lies entirely within } A \\ &\Longrightarrow \quad \text{some neighbourhood of } x \text{ fails to intersect } X - A \\ &\Longrightarrow \quad x \notin \operatorname{Cl}(X - A) \\ &\Longrightarrow \quad x \notin \operatorname{Bd} A. \end{aligned}$$

37. To establish the inclusion $\operatorname{Int} A \cup \operatorname{Bd} A \subset \operatorname{Cl} A$, we need only note that

$$\operatorname{Int} A \subset A \subset \operatorname{Cl} A, \qquad \operatorname{Bd} A = \operatorname{Cl} A \cap \operatorname{Cl}(X - A) \subset \operatorname{Cl} A.$$

To establish the reverse inclusion, suppose that $x \in \operatorname{Cl} A$. Then every neighbourhood of x must intersect A and we consider two cases.

- If some neighbourhood of x intersects A but not its complement $X - A$, then that neighbourhood lies entirely within A, so x is in the interior of A by definition.
- If every neighbourhood of x intersects both A and its complement $X - A$, then x is in the boundary of A by definition.

In either case then, the inclusion $\operatorname{Cl} A \subset \operatorname{Int} A \cup \operatorname{Bd} A$ follows.

38. To see that $X - \operatorname{Cl} A = \operatorname{Int}(X - A)$, we note that

$$\begin{aligned} x \notin \operatorname{Cl} A &\iff \text{some neighbourhood of } x \text{ fails to intersect } A \\ &\iff \text{some neighbourhood of } x \text{ lies entirely within } X - A \\ &\iff x \in \operatorname{Int}(X - A). \end{aligned}$$

39. If A is compact, then A is compact in a Hausdorff space, hence also closed. If A is closed, then A is closed in a compact space, hence also compact.

40. Let n be a positive integer. Being smaller than the least upper bound, $\sup A - 1/n$ is not an upper bound of A. In particular, there exists some $x_n \in A$ such that

$$\sup A - \frac{1}{n} < x_n \leq \sup A.$$

- If equality happens to hold for some n, then $\sup A = x_n$ is a point of A.
- If strict inequality holds for all n, then we have a sequence $\{x_n\}$ of points in A such that $x_n \to \sup A$, yet $x_n \neq \sup A$ for all n. This makes $\sup A$ a limit point of A.

Since $\sup A$ is either a point of A or a limit point of A, we deduce that $\sup A \in \operatorname{Cl} A$.

► Suppose now that $B \subset \mathbb{R}$ is compact. By the Heine-Borel theorem, B is then closed and bounded. Since B is bounded, its closure must contain $\sup B$ by above. Since B is closed, however, it is equal to its own closure. This means that B must contain its supremum.

41. Given any point $y \in A$, we have $x \neq y$. Since X is Hausdorff, we may thus find disjoint open sets $U(y)$ and $V(y)$ containing y and x, respectively. Since the sets $U(y)$ form an open cover of A, finitely many of them do. Say $A \subset U(y_1) \cup \cdots \cup U(y_n)$ and let

$$U = U(y_1) \cup \cdots \cup U(y_n), \qquad V = V(y_1) \cap \cdots \cap V(y_n).$$

Then U and V are open sets containing A and x, respectively. Moreover, we have

$$\begin{aligned} z \in U \quad &\implies \quad z \in U(y_i) \quad \text{for some } i \\ &\implies \quad z \notin V(y_i) \quad \text{for some } i \\ &\implies \quad z \notin V. \end{aligned}$$

In particular, U and V are also disjoint, as needed.

42. Given any point $x \in B$, we have $x \in X - A$ and so the previous problem allows us to find disjoint open sets $U(x)$ and $V(x)$ containing A and x, respectively. Since the sets $V(x)$ form an open cover of B, finitely many of them do. Say $B \subset V(x_1) \cup \cdots \cup V(x_n)$ and let

$$U = U(x_1) \cap \cdots \cap U(x_n), \qquad V = V(x_1) \cup \cdots \cup V(x_n).$$

Then U and V are open sets containing A and B, respectively. Moreover, we have

$$\begin{aligned} z \in U \quad &\implies \quad z \in U(x_i) \quad \text{for each } i \\ &\implies \quad z \notin V(x_i) \quad \text{for each } i \\ &\implies \quad z \notin V. \end{aligned}$$

In particular, U and V are also disjoint, as needed.

43. Suppose that U is closed in X. Being closed in a compact space, U is then compact. We know that the continuous image of a compact set is compact; so $f(U)$ is compact as well. Being compact in a Hausdorff space, $f(U)$ must then be closed.

44. Being compact in a Hausdorff space, A is closed in X. This makes $A \cap B$ closed in B by the definition of the subspace topology. Being closed in a compact space, $A \cap B$ is then compact itself.

45. Suppose the intersection of the C_i's is empty. According to De Morgan's law then,

$$\bigcup_{i=1}^{\infty} (X - C_i) = X - \bigcap_{i=1}^{\infty} C_i = X,$$

so the sets $X - C_i$ form an open cover of X. By compactness, finitely many of these sets must cover X; suppose the first n do. Using De Morgan's law, we now find

$$X - C_n = X - \bigcap_{i=1}^{n} C_i = \bigcup_{i=1}^{n} (X - C_i) = X,$$

which is impossible since $C_n \neq \varnothing$. Thus, the intersection of the C_i's cannot be empty.

46. Being continuous, the restriction $g\colon [a,b] \to \mathbb{R}$ does have the intermediate value property. Moreover, $g(a)$ and $g(b)$ have opposite signs by assumption; namely, one of them is positive and the other one is negative. This also implies that $g(c) = 0$ for some $c \in (a,b)$.

47. Note that $a \le g(a)$ and $g(b) \le b$ by assumption. If either of these inequalities happens to be an equality, the result follows trivially. Suppose now that $a < g(a)$ and $g(b) < b$. Then the function $f(x) = g(x) - x$ is continuous with

$$f(a) = g(a) - a > 0, \qquad f(b) = g(b) - b < 0.$$

By the intermediate value property, there must exist some $c \in (a, b)$ such that $f(c) = 0$. Since this actually implies that $g(c) = c$, the proof is complete.

48. Suppose A is a finite subset of $\mathbb{R}^2$ and let x, y be points in the complement of A. Since there are infinitely many lines passing through x, we can always find a line through x that fails to intersect A. Now, follow this line until you reach a point z and then follow the straight line from z to y. Since there are infinitely many points z at which you can stop before making a turn, one of the resulting paths fails to intersect A. That would also be a path from x to y which lies entirely in the complement of A.

49a. Not compact; not connected; not path-connected.

49b. Not compact; connected; path-connected.

49c. Compact; connected; path-connected.

50. First, we use induction to show that the union $B_n = A_1 \cup \cdots \cup A_n$ of the first n sets is connected. When $n = 1$, we have $B_1 = A_1$ and this set is connected by assumption. Suppose now that B_n is connected for some n. Since B_n contains A_n, it must have a point in common with A_{n+1}. Since B_n and A_{n+1} are both connected, their union B_{n+1} is thus connected as well. In particular, all the B_n's are connected by induction.

▶ Next, we note that the B_n's have a point in common because they all contain A_1. This means that their union must also be connected. Since the union of the B_n's coincides with the union of the A_n's, the proof is complete.

51. Being restrictions of continuous maps, both $\widetilde{f}$ and its inverse are continuous. As they are also bijective, we conclude that $\widetilde{f}$ is a homeomorphism.

▶ Suppose now that we have a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^2$. Then the restriction

$$\widetilde{f}\colon \mathbb{R} - \{0\} \to \mathbb{R}^2 - \{f(0)\}$$

is a homeomorphism as well. Note that the domain is not path-connected, as it is not even connected. On the other hand, the image is path-connected in view of Problem 48. Since this is a contradiction, no homeomorphism exists between $\mathbb{R}$ and $\mathbb{R}^2$.

52. Suppose that X is path-connected and $f\colon X \to Y$ is continuous. Let $f(x_0), f(x_1)$ be any two points in the image. We know there exists a path $\gamma\colon [0,1] \to X$ with

$$\gamma(0) = x_0, \qquad \gamma(1) = x_1.$$

Since f is continuous, the composition $f \circ \gamma\colon [0,1] \to Y$ is then a path with

$$(f \circ \gamma)(0) = f(x_0), \qquad (f \circ \gamma)(1) = f(x_1).$$

53a. As we already know, the closure of a connected set is always connected. This is one of the two "marginally useful" facts: the inclusion of limit points does not ruin connectedness.

53b. The boundary of a connected set does not have to be connected. For instance, $[0,1]$ is connected because it is an interval, yet its boundary $\{0,1\}$ is not connected because it is not an interval.

53c. The interior of a connected set does not have to be connected; see Problem 49.

54. Suppose $C|D$ is a partition of $Y \cup A$. Being a connected subset of this partition, Y must then lie within either C or D. Assume $Y \subset C$ without loss of generality. Then it must be the case that $D \subset A$, since

$$\begin{aligned} x \in D \quad &\Longrightarrow \quad x \in Y \cup A \quad \text{yet } x \notin C \\ &\Longrightarrow \quad x \in Y \cup A \quad \text{yet } x \notin Y \\ &\Longrightarrow \quad x \in A. \end{aligned}$$

Now, consider the sets $B \cup C$ and D. These are nonempty, disjoint and their union is

$$B \cup C \cup D = B \cup A \cup Y = (X - Y) \cup Y = X.$$

If we can also show that they are open in X, then $B \cup C|D$ would be a partition of X. This would violate the connectedness of X and would also complete the proof.

► To check that $B \cup C$ is open, it suffices to check that its complement D is closed. Thus, it suffices to check that $\operatorname{Cl} D = D$. Using the properties of closures, we get

$$D \subset A \quad \Longrightarrow \quad \operatorname{Cl} D \subset \operatorname{Cl} A.$$

Besides, one of the "marginally useful" facts suggests that $\operatorname{Cl} A$ does not intersect B, so

$$\operatorname{Cl} D \subset \operatorname{Cl} A \subset X - B = C \cup D.$$

Since the very same fact ensures that $\operatorname{Cl} D$ does not intersect C, this actually implies

$$\operatorname{Cl} D \subset D.$$

As the reverse inclusion $D \subset \operatorname{Cl} D$ is always true, we deduce that $\operatorname{Cl} D = D$.

► To check that D is open, we similarly check that $\operatorname{Cl}(B \cup C) = B \cup C$. In this case,

$$\operatorname{Cl}(B \cup C) = \operatorname{Cl} B \cup \operatorname{Cl} C \subset (X - A) \cup (X - D)$$

because $\operatorname{Cl} B$ fails to intersect A and $\operatorname{Cl} C$ fails to intersect D. Thus, we have

$$\operatorname{Cl}(B \cup C) \subset (B \cup Y) \cup (B \cup C) \subset B \cup C$$

because $Y \subset C$ by above. Once again, this implies $\operatorname{Cl}(B \cup C) = B \cup C$, as needed.