Maths 212, Homework #9
First four problems:
due Thursday, Jan. 26

63. Show that the sequence $f_n(x) = x^n(1-x)$ converges uniformly on $[0,1]$.

64. Show that the function $f(x) = 1/x$ is uniformly continuous on $(1,2)$.

65. Show that the function $f(x) = 1/x$ is not uniformly continuous on $(0,1)$.

66. Let $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions such that

$$f_n \to f \quad \text{uniformly on } X.$$

Given a sequence of points $x_n \in X$ with $x_n \to x$, show that $f_n(x_n) \to f(x)$ as well.

67. Let $f_n \colon X \to \mathbb{R}$ be a sequence of bounded functions such that $f_n \to f$ uniformly on X. Show that f is bounded as well.

68. Let $f_n, g_n \colon X \to \mathbb{R}$ be sequences of bounded functions such that

$$f_n \to f \quad \text{uniformly on } X, \qquad g_n \to g \quad \text{uniformly on } X.$$

Show that $f_n g_n \to fg$ uniformly on X as well.

69. Let $f \colon X \to \mathbb{R}$ be uniformly continuous on X and let $\{x_n\}$ be a Cauchy sequence in X. Show that the sequence $\{f(x_n)\}$ is also Cauchy.

70. Show that the sequence $f_n(x) = \dfrac{x}{1+nx^2}$ is uniformly convergent on $\mathbb{R}$.

71. Let $f_n \colon X \to \mathbb{R}$ be a sequence of uniformly continuous functions such that

$$f_n \to f \quad \text{uniformly on } X.$$

Show that f is uniformly continuous on X as well.

Some Hints

63. Show that $f_n(x)$ attains its maximum at $x = n/(n+1)$. You will also need to know that

$$\left(\frac{n+1}{n}\right)^n \to e.$$

64. Given points x, y in the interval $(1, 2)$, one has

$$|f(x) - f(y)| = \frac{|x-y|}{xy} \leq |x-y|.$$

65. Letting $x_n = 1/n$ and $y_n = 1/(n+1)$, one has $|f(x_n) - f(y_n)| = 1$.

66. Note that $f(x_n) \to f(x)$ by continuity and then use the triangle inequality

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|.$$

67. Use the triangle inequality $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$.

68. Use the previous problem together with the triangle inequality

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)|.$$

69. Write the definition of uniform continuity in the form

$$|f(x_n) - f(x_m)| < \varepsilon \qquad \text{whenever } \ d(x_n, x_m) < \delta.$$

70. Show that $f_n(x)$ attains its maximum at $x = 1/\sqrt{n}$.

71. Use the triangle inequality

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$