Maths 212, Homework #7
To be discussed in class:
Thursday, Dec. 8

46. (Bolzano's theorem) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous with

$$f(a) \cdot f(b) < 0 \qquad \text{for some real numbers } \ a < b.$$

Show that there exists a real number $c \in (a, b)$ such that $f(c) = 0$.

47. (Fixed point theorem) Suppose that $g\colon [a, b] \to [a, b]$ is continuous. Show that there exists a real number $c \in [a, b]$ such that $g(c) = c$.

48. Show that the complement of a finite subset of $\mathbb{R}^2$ is path-connected.

49. Which of the following subsets of $\mathbb{R}^2$ are compact? connected? path-connected?

    (a) $B_1((-1, 0)) \cup B_1((1, 0))$
    (b) $B_1((-1, 0)) \cup \mathrm{Cl}(B_1((1, 0)))$
    (c) $\mathrm{Cl}(B_1((-1, 0))) \cup \mathrm{Cl}(B_1((1, 0)))$

50. Suppose $A_1, A_2, \ldots$ are connected subsets of a topological space X such that

$$A_n \cap A_{n+1} \neq \varnothing$$

for each integer $n \geq 1$. Show that the union of these sets is also connected.

51. Given a homeomorphism $f\colon X \to Y$ and a point $x \in X$, show that the restriction

$$\widetilde{f}\colon X - \{x\} \to Y - \{f(x)\}$$

is also a homeomorphism. Using this fact, show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

52. Show that the continuous image of a path-connected space is path-connected itself.

53. Suppose A is a connected subset of a topological space X. Does the closure of A have to be connected as well? How about the boundary? How about the interior?

54. Suppose $Y \subset X$ are both connected and $A|B$ is a partition of $X - Y$. Show that $Y \cup A$ is connected as well.