Maths 212, Homework #2

First three problems:

due Thursday, Oct. 27

8. Let $\{X, d\}$ be a metric space and define
$$d_0(x, y) = \min\{1, d(x, y)\}$$
for all $x, y \in X$. Prove that d_0 is a metric as well.

9. The Cartesian product $X \times Y$ of two sets is defined to be the set
$$X \times Y = \{(x, y) \ : \ x \in X, \ y \in Y\}.$$
Assuming that $\{X, d_1\}$ and $\{Y, d_2\}$ are metric spaces, show that $\{X \times Y, d\}$ is also a metric space when the distance is defined by
$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}.$$

10. Let $\{X, d\}$ be a metric space. Suppose the points $x, y \in X$ are distinct and let
$$r = d(x, y)$$
denote their distance. Show that the open balls $B_{r/2}(x)$ and $B_{r/2}(y)$ are disjoint, i.e.,
$$B_{r/2}(x) \cap B_{r/2}(y) = \varnothing.$$

11. Let $\{X, d\}$ be a metric space. Given any point $x \in X$ and any real number $r \geq 0$, show that the set $U = \{y \in X \ : \ d(x, y) > r\}$ is open in X.

12. Consider the Euclidean space $\mathbb{R}^2$ together with the metric
$$d_\infty((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$
Draw a sketch of the open ball $B_1((0, 0))$ of radius 1 around the origin.

13. Let $\{X, d\}$ be a discrete metric space. Show that all subsets of X are open in X.

14. Consider the Euclidean space $\mathbb{R}^2$. Let d_∞ denote the metric of the previous exercise, let d_1 denote the ℓ^1-metric
$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$
and let d_2 denote the Euclidean ℓ^2-metric
$$d_2((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$
Show that these metrics are all Lipschitz equivalent.