Maths 212, Homework #17
First three problems:
due Thursday, Apr. 20

125. Define an equivalence relation on the plane $X = \mathbb{R}^2$ by setting
$$(x_1, y_1) \sim (x_2, y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2.$$
Show that the quotient space $\overline{X} = X/{\sim}$ is homeomorphic to $[0, \infty)$.

126. Show that the composition of two quotient maps is a quotient map itself.

127. Let $X = B^2$ be the unit disc and consider the equivalence relation $\sim$ which identifies all points on the boundary. What is the quotient space $\overline{X} = X/{\sim}$ homeomorphic to?

128. Give a formal proof that justifies your answer in the previous problem.

129. Define an equivalence relation on $X = \mathbb{R}$ by setting
$$x \sim y \iff x = \lambda y \text{ for some } \lambda > 0.$$
Show that the quotient space $\overline{X} = X/{\sim}$ is not Hausdorff.

130. Show that every retraction is a quotient map.

131. Define an equivalence relation on $X = S^2$ by setting
$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff z_1 = z_2.$$
Show that the quotient space $\overline{X} = X/{\sim}$ is homeomorphic to $[-1, 1]$.

Some Hints

125. Define a function $f\colon X \to [0, \infty)$ by the formula
$$f(x, y) = x^2 + y^2.$$
Using one of our lemmas, you will then get a continuous function $\overline{f}\colon \overline{X} \to [0, \infty)$. Using Problem 43, show that this is actually a homeomorphism.

126. Assuming that $f\colon X \to Y$ and $g\colon Y \to Z$ are both quotient maps,
$$U \text{ is open in } Z \iff g^{-1}(U) \text{ is open in } Y$$
and also
$$V \text{ is open in } Y \iff f^{-1}(V) \text{ is open in } X.$$

127. Take a circular piece of paper and glue its boundary together. What familiar space does this give rise to?

128. Let $X = B^2$ and let Y be the quotient space obtained in the previous problem. You will need to find a continuous function $f\colon X \to Y$ which is constant throughout the boundary. Using one of our lemmas, you may then get a continuous function $\overline{f}\colon \overline{X} \to Y$.

129. First, show that the quotient space $\overline{X}$ consists of three elements only, namely
$$\overline{X} = \{[0], [1], [-1]\}.$$
Which subsets of $\overline{X}$ are open? Do distinct points have disjoint neighbourhoods?

130. Let $r\colon X \to A$ be the retraction and $i\colon A \to X$ be the inclusion. Show that
$$U = r^{-1}(U) \cap A$$
for all subsets $U \subset A$.

131. Consider the function $f\colon X \to [-1, 1]$ defined by $f(x, y, z) = z$. Using one of our lemmas, you will get a continuous function $\overline{f}\colon \overline{X} \to [-1, 1]$.