Maths 212, Homework #15
First four problems:
due Thursday, Mar. 9

109. Suppose that X is path connected and let $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if $\hat{\alpha} = \hat{\beta}$ for all paths α, β from x_0 to x_1.

110. Suppose that X has the discrete topology. Show that the projection $p\colon X \times Y \to Y$ is a covering map for any space Y whatsoever.

111. Show that every covering space of a Hausdorff space must also be Hausdorff.

112. Letting B^2 denote the unit ball in $\mathbb{R}^2$, show that there is no retraction $r\colon B^2 \to S^1$.

113. Show that homeomorphic spaces have isomorphic fundamental groups. Namely, assuming

$$f\colon (X, x_0) \to (Y, y_0)$$

is a homeomorphism between topological spaces, show that the induced homomorphism

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is actually an isomorphism between fundamental groups.

114. Suppose X is connected and $p\colon Y \to X$ is a covering map. Assuming that $p^{-1}(x_0)$ has n elements for some $x_0 \in X$, show that $p^{-1}(x)$ has n elements for all $x \in X$.

115. Suppose that X is simply connected and let Y be a path connected covering space of X. Show that the covering map $p\colon Y \to X$ is actually a homeomorphism.

116. Show that a retract of a contractible space must also be contractible.

Some Hints

109. In the case that $\pi_1(X, x_0)$ is abelian, $[\alpha * \overline{\beta}]$ commutes with any loop $[f]$ around x_0, hence

$$[f] * [\alpha] * [\overline{\beta}] = [f] * [\alpha * \overline{\beta}] = [\alpha * \overline{\beta}] * [f] = [\alpha] * [\overline{\beta}] * [f].$$

Use this fact together with the definition of $\hat{\alpha}$ to prove the first part. For the second part, you have to show that

$$[f] = [h] * [f] * [\overline{h}]$$

for all loops f, h around x_0; use the fact that $\hat{\alpha}$ is an isomorphism.

110. Note that $p^{-1}(U)$ is the union of sets of the form $V_y = X \times \{y\}$.

111. Suppose $y_0 \neq y_1$ and consider two cases. If $p(y_0)$ happens to coincide with $p(y_1)$, then this point has a neighbourhood U such that $p^{-1}(U)$ is a disjoint union of V_α's; argue that the points y_0, y_1 lie in distinct V_α's. The case $p(y_0) \neq p(y_1)$ is even easier.

112. Letting i denote inclusion, we have maps

$$i\colon S^1 \to B^2, \qquad r\colon B^2 \to S^1$$

and induced homomorphisms

$$i_*\colon \pi_1(S^1, y_0) \to \pi_1(B^2, x_0), \qquad r_*\colon \pi_1(B^2, x_0) \to \pi_1(S^1, y_0).$$

Note that one of the fundamental groups is trivial, while $r \circ i$ is the identity map.

113. Let g be the inverse of f and note that $g_* \circ f_* = (g \circ f)_*$ is the identity map.

114. Let A be the set of all $x \in X$ for which $p^{-1}(x)$ has n elements. Try to show that A is both open and closed in X. Since X is connected, this also implies that $A = X$.

115. In this case, there is a bijective map $\Phi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$.

116. Let $r\colon X \to Y$ be a retraction and let $F\colon X \times I \to X$ be a homotopy between the identity and a constant map. You need only define a homotopy $G\colon Y \times I \to Y$ between the identity and a constant map.