Maths 212, Homework #14
First four problems:
due Thursday, Mar. 2

102. Let p, q be positive numbers with $1/p + 1/q = 1$. Fix some element $a = (a_1, a_2, \ldots)$ of ℓ^q and define the operator $T\colon \ell^p \to \mathbb{R}$ by setting

$$Tx = \sum_{i=1}^{\infty} a_i\, x_i \qquad \text{for each } x = (x_1, x_2, \ldots) \in \ell^p.$$

Show that the norm of this operator is equal to $||T|| = ||a||_q$.

103. Show that every contractible space is path connected.

104. Suppose $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ are continuous. Assuming that f_0 is homotopic to f_1 and g_0 homotopic to g_1, show that $g_0 \circ f_0$ is homotopic to $g_1 \circ f_1$.

105. Let α be a path in X from x to y, and let β be a path in X from y to z. Show that

$$\gamma = \alpha * \beta \quad \Longrightarrow \quad \hat{\gamma} = \hat{\beta} \circ \hat{\alpha}.$$

106. Suppose that X is a convex subset of $\mathbb{R}^n$ in the sense that

$$x, y \in X \quad \Longrightarrow \quad \lambda x + (1 - \lambda) y \in X \qquad \text{for each } 0 \le \lambda \le 1.$$

Show that any two paths γ_0, γ_1 in X which have the same endpoints are path homotopic.

107. Given a continuous function $f\colon X \to Y$ and a point $x_0 \in X$, show that the induced map

$$f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

defined by $f_*([h]) = [f \circ h]$ is a group homomorphism.

108. Let $A \subset X$ and suppose that $r\colon X \to A$ is a retraction. Show that the induced map

$$r_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(A, x_0)$$

is surjective for each $x_0 \in A$.

Some Hints

102. First, use Hölder's inequality to show that $||T|| \leq ||a||_q$. To show that $||T|| \geq ||a||_q$, set

$$x_i = |a_i|^{q/p} \cdot \operatorname{sign} a_i$$

and show that $x = (x_1, x_2, \ldots)$ is an element of ℓ^p such that $Tx = ||a||_q \cdot ||x||_p$. Here, the sign factor is included so that $a_i x_i = |a_i|^{q/p+1}$ for each i.

103. Let a, b be any two points in X and let $F\colon X \times I \to X$ be a homotopy between the identity function and the constant map $f(x) = b$. You need to find a path from a to b.

104. Let $F\colon X \times I \to Y$ be a homotopy between the f_i's and $G\colon Y \times I \to Z$ be a homotopy between the g_i's. What properties does the function $H(x,t) = G(F(x,t),t)$ have?

105. Using the definition $\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$, try to simplify the expression $\hat{\beta}(\hat{\alpha}([f]))$.

106. Use a straight-line path homotopy.

107. It suffices to show that $f \circ (g * h) = (f \circ g) * (f \circ h)$.

108. Note that the inclusion $i\colon A \to X$ is continuous and that $i \circ r$ is the identity map on A.