Maths 212, Homework #13
First four problems:
due Thursday, Feb. 23

95. Prove Minkowski's inequality. In other words, show that the inequality

$$\left(\sum_i |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |y_i|^p\right)^{1/p}$$

holds for all real numbers x_i, y_i and each $p > 1$.

96. Show that ℓ^p is complete for each $1 \leq p < \infty$.

97. Let $a = (a_1, a_2, \ldots)$ be a bounded sequence of real numbers and set

$$Tx = (a_1 x_1, a_2 x_2, \ldots) \qquad \text{for each } x = (x_1, x_2, \ldots) \in \ell^2.$$

Show that T is a bounded linear transformation $T\colon \ell^2 \to \ell^2$ and compute its norm $||T||$.

98. Let ℓ_0 consist of all sequences of real numbers that contain only finitely many nonzero terms. Show that ℓ_0 is dense in ℓ^p for each $1 \leq p < \infty$.

99. Let C_0 consist of all sequences of real numbers $x = (x_1, x_2, \ldots)$ such that

$$\lim_{n\to\infty} x_n = 0.$$

Show that C_0 is complete with respect to the metric induced by the ℓ^∞-norm.

100. Let ℓ_0 be as in Problem 98 and set

$$Tx = \sum_{n=1}^{\infty} x_n \qquad \text{for each } x = (x_1, x_2, \ldots) \in \ell_0.$$

(a) Show that $T\colon \ell_0 \to \mathbb{R}$ is continuous when ℓ_0 is equipped with the ℓ^1-norm.

(b) Show that $T\colon \ell_0 \to \mathbb{R}$ is not continuous when ℓ_0 is equipped with the ℓ^2-norm.

101. Show that the averaging operator $A\colon \ell^\infty \to \ell^\infty$ defined by

$$Ax = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right)$$

is a bounded linear transformation and find its norm.

Some Hints

95. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ for convenience. Try to show that

$$\begin{aligned} ||x+y||_p^p &= \sum_i |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &\leq \sum_i |x_i| \cdot |x_i + y_i|^{p-1} + \sum_i |y_i| \cdot |x_i + y_i|^{p-1} \\ &\leq ||x||_p \cdot ||x+y||_p^{p-1} + ||y||_p \cdot ||x+y||_p^{p-1} \end{aligned}$$

using the triangle and Hölder's inequality. Conclude that $||x+y||_p \leq ||x||_p + ||y||_p$.

96. Suppose that $x_n = (x_{n1}, x_{n2}, \ldots)$ is a Cauchy sequence in ℓ^p, deduce that x_{nk} is a Cauchy sequence in $\mathbb{R}$ for each k, and then finish the proof.

97. Linearity is rather easy to establish. For the remaining parts, you will need to note that

$$\sum_{i=1}^{\infty} a_i^2 x_i^2 \leq ||a||_\infty^2 \sum_{i=1}^{\infty} x_i^2 \quad \Longrightarrow \quad ||Tx||_2 \leq ||a||_\infty ||x||_2.$$

This leads to the inequality $||T|| \leq ||a||_\infty$. Try to argue that equality actually holds; this is not a trivial thing to do, but it's not terribly hard, either.

98. Pick an element $a = (a_1, a_2, \ldots)$ in ℓ^p and try to find a sequence of elements $x_n \in \ell_0$ such that $||x_n - a||_p \to 0$. This actually shows that $\mathrm{Cl}\, \ell_0 = \ell^p$, as needed. In case you get stuck, we did something similar in class; see Example 4.

99. Suppose that $x_n = (x_{n1}, x_{n2}, \ldots)$ is a Cauchy sequence in C_0, deduce that x_{nk} is a Cauchy sequence in $\mathbb{R}$ for each k, and then finish the proof.

100. To say that T is continuous is to say that T is bounded.

101. Linearity is rather easy to establish. For the remaining parts, you will need to note that

$$|x_1 + \ldots + x_n| \leq n \cdot ||x||_\infty \quad \Longrightarrow \quad ||Ax||_\infty \leq ||x||_\infty.$$

Can you find some $x \in \ell^\infty$ such that equality holds in the inequality above?