Maths 212, Homework #12
First four problems:
due Thursday, Feb. 16

89. Fix some function $f\colon \mathbb{R} \to \mathbb{R}$ and define a sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ by setting

$$f_n(x) = f(nx) \qquad \text{for each } x \in \mathbb{R}.$$

Assuming that the f_n's form an equicontinuous family, show that f must be constant.

90. Suppose that $f_n\colon [0,1] \to \mathbb{R}$ is a sequence of continuous functions such that

$$|f_n(x)| \leq 1 \qquad \text{for all } 0 \leq x \leq 1 \text{ and each } n \geq 1.$$

Define a sequence of functions $F_n\colon [0,1] \to \mathbb{R}$ by setting

$$F_n(x) = \int_0^x f_n(s)\,ds \qquad \text{for all } 0 \leq x \leq 1 \text{ and each } n \geq 1.$$

Show that F_n has a subsequence which converges uniformly on $[0,1]$.

91. Suppose that X is compact and let $f_n\colon X \to \mathbb{R}$ be a sequence of equicontinuous functions such that $f_n \to f$ pointwise on X. Show that the convergence is actually uniform.

92. Suppose that $f_n\colon [0,1] \to \mathbb{R}$ is a sequence of continuous functions such that

$$f_n \to f \qquad \text{uniformly on } [0,1].$$

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

93. Show that the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by the formula

$$f_n(x) = \frac{2nx}{1+n^2x^2}$$

do not form an equicontinuous family at the origin.

94. Let F be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \leq |x-y| \qquad \text{for all } 0 \leq x, y \leq 1$$

as well as $f(0) = 0$. Show that F is compact.

Some Hints

89. Note that $|f_n(a/n) - f_n(0)| = |f(a) - f(0)|$ for all $0 \le a \le 1$ and each $n \ge 1$.

90. Use the corollary to the Arzela-Ascoli theorem.

91. Note that equicontinuity is really uniform equicontinuity here. To show that the sequence is uniformly convergent, you need only show it is uniformly Cauchy.

92. Establish the inequality

$$\left| \int_0^1 f_n(x)\, dx - \int_0^1 f(x)\, dx \right| \le d_\infty(f_n, f).$$

93. Note that $|f_n(1/n) - f_n(0)| = 1$ for each $n \ge 1$.

94. We did something similar in class.