Maths 212, Homework #11
First five problems:
due Thursday, Feb. 9

80. Find a nested sequence of nonempty, bounded subsets of $\mathbb{R}$ whose intersection is empty.

81. Find a nested sequence of nonempty, closed subsets of $\mathbb{R}$ whose intersection is empty.

82. Is an infinite discrete metric space bounded? Is it totally bounded? Explain.

83. Given a uniformly continuous function f on a totally bounded metric space X, show that the image $f(X)$ is totally bounded as well.

84. Does the sequence $f_n(x) = (x+n)^2$ form an equicontinuous family at $x = 0$? Explain.

85. Show that the sequence $f_n(x) = \cos(nx)$ does not form an equicontinuous family at $x = 0$.

86. Suppose that X is compact Hausdorff and let $f\colon X \to X$ be continuous. Show that there exists a closed subset $A \subset X$ such that $f(A) = A$.

87. Given a metric space X and some nonempty subset $A \subset X$, consider the function

$$\rho(x) = \inf_{z \in A} d(x, z).$$

(a) Show that $x \in \operatorname{Cl} A$ if and only if $\rho(x) = 0$.

(b) Show that ρ is Lipschitz continuous.

88. Show that the zero function is the only function $f \in \mathcal{C}([0,1])$ such that

$$f(x) = \int_0^x (x-y)\, f(y)\, dy \qquad \text{for all } 0 \le x \le 1.$$

Some Hints

80. Look for a nested sequence of open subsets.

81. Look for a nested sequence of unbounded subsets.

82. Yes; no.

83. You need only show that every sequence in $f(X)$ has a Cauchy subsequence. Problem 69 should be of help here, still do not use it without proof.

84. Show that $|f_n(1/n) - f_n(0)| > 2$ for all n.

85. Show that $|f_n(\pi/n) - f_n(0)| = 2$ for all n.

86. Let $A_1 = X$ and set $A_{n+1} = f(A_n)$ for each $n \geq 1$. This is a nested sequence of compact sets; complete the proof by looking at their nonempty intersection.

87a. To say that $x \in \operatorname{Cl} A$ is to say that some sequence of points of A converges to x.

87b. Letting $x, y \in X$ be fixed, show that $\rho(x) \leq d(x, y) + d(y, z)$ for all $z \in A$.

88. Show that the given integral defines a contraction on $\mathcal{C}([0, 1])$ with the d_∞ metric.