Maths 212, Homework #10
First four problems:
due Thursday, Feb. 2

72. Show that every discrete metric space is complete.

73. Let X be an arbitrary metric space. Show that the union of two complete subsets of X is complete as well.

74. Let $A = \mathcal{C}([0,1])$ be the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ with the d_∞ metric. Show that A is complete.

75. Let $A = \mathcal{P}([0,1])$ be the space of polynomial functions $p\colon [0,1] \to \mathbb{R}$ with the d_∞ metric. Show that A is not complete.

76. Show that the function $f(x) = x^2/3$ is a contraction on $(0,1)$ even though it has no fixed point. Why doesn't this violate Banach's fixed point theorem?

77. Let $h(x)$ be a fixed function in $X = \mathcal{C}([0,1/2])$ and define $T\colon X \to X$ by the formula
$$T(f)(x) = h(x) + \int_0^x f(t)\,dt.$$
Show that T has a fixed point in X.

78. Suppose $f\colon [0,1] \to \mathbb{R}$ is a differentiable function such that f' is continuous. Show that f is Lipschitz continuous as well.

79. Show that the continuous image of a complete metric space need not be complete.

Some Hints

72. What kinds of sequences are Cauchy in the discrete metric?

73. Let A, B be the two subsets and consider a Cauchy sequence in $A \cup B$. Argue that some subsequence lies entirely within either A or B. Why is this good enough?

74. Show that A is contained in $\mathscr{B}([0,1],\mathbb{R})$ and that A is closed in this space. To prove the latter statement, you will need the limit of continuous functions to be continuous. This is not true, in general; why is it true here?

75. Show that A is contained, but not closed, in $\mathcal{C}([0,1])$. To prove the latter statement, try to find a sequence of polynomials that converges to e^x. Why is e^x not a polynomial?

76. Given points x, y in the unit interval $(0,1)$, one has
$$|x^2 - y^2| = |x+y| \cdot |x-y| \leq 2 \cdot |x-y|.$$

77. Show that $d_\infty(T(f), T(g)) \leq d_\infty(f,g)/2$ and then use Banach's fixed point theorem.

78. Show that f' is bounded and then use the fundamental theorem of calculus to write
$$f(x) - f(y) = \int_y^x f'(s)\, ds.$$

79. Any homeomorphism between $\mathbb{R}$ and an open interval will serve as a counterexample.